## E148: A system of two qubits, perturbation theory

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## The problem:

Qubit is a device with two basis states $|x=0\rangle,|x=1\rangle$. Assume that both states have the same energy and that the jumping amplitude between them is $c$. We are given a system of two qubits with basis states $\left|x_{a}, x_{b}\right\rangle$. It turns out that when the two qubits are in the state $x=1$, there is an energetic payment $u$. The hamiltonian of the system can be written as $\mathcal{H}=\mathcal{H}_{0}+u V$.
(1) Write the matrix of $V$ in the standard basis (no algebraic work is needed).

The hamiltonian $\mathcal{H}_{0}$ is diagonal in a certain basis. Indicate the basis states as $|S S\rangle,|S A\rangle,|A S\rangle,|A A\rangle$.
(2) Write column vectors which represent the new basis states in the standard basis.
(3) Write the diagonal matrix which represents $\mathcal{H}_{0}$ in the basis you defined (no algebraic work is needed).
(4) Write in the new basis the perturbation matrix $V$.
(5) Find the energy eigenvalues of the system up to the first order in the interaction.
(6) Find the ground state energy up to the second order in the interaction.

## The solution:

(1) The states of a single qubit are:

$$
|0\rangle=\binom{1}{0} \quad|1\rangle=\binom{0}{1}
$$

With the appropriate hamiltonian:

$$
\tilde{\mathcal{H}}=\left(\begin{array}{ll}
0 & c \\
c & 0
\end{array}\right)
$$

We construct the basis of the composite system using the tensor product $\left|x_{a}, x_{b}\right\rangle=\left|x_{a}\right\rangle \otimes\left|x_{b}\right\rangle$ on the single qubit states:

$$
|0,0\rangle=\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right) \quad|0,1\rangle=\left(\begin{array}{l}
0 \\
1 \\
0 \\
0
\end{array}\right) \quad|1,0\rangle=\left(\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right) \quad|1,1\rangle=\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right)
$$

The effect of perturbation is only on the $|1,1\rangle$ state, meaning a change in this state's energy by $u$. Therefore the perturbation matrix in the standard basis is:

$$
V=u|1,1\rangle\langle 1,1|=\left(\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & u
\end{array}\right)
$$

(2) The unperturbed hamiltonian for a single qubit is diagonal in the symmetric and antisymmetric basis:

$$
|S\rangle=\frac{1}{\sqrt{2}}\binom{1}{1} \quad|A\rangle=\frac{1}{\sqrt{2}}\binom{1}{-1}
$$

From here we obtain the basis for the composite system, in the same method as in section (1):

$$
|S S\rangle=\frac{1}{2}\left(\begin{array}{l}
1 \\
1 \\
1 \\
1
\end{array}\right) \quad|S A\rangle=\frac{1}{2}\left(\begin{array}{c}
1 \\
-1 \\
1 \\
-1
\end{array}\right) \quad|A S\rangle=\frac{1}{2}\left(\begin{array}{c}
1 \\
1 \\
-1 \\
-1
\end{array}\right) \quad|A A\rangle=\frac{1}{2}\left(\begin{array}{c}
1 \\
-1 \\
-1 \\
1
\end{array}\right)
$$

(3) The unperturbed hamiltonian in the new basis is:

$$
\begin{aligned}
& \mathcal{H}_{0}=\tilde{\mathcal{H}} \otimes 1_{2 \times 2}+1_{2 \times 2} \otimes \tilde{\mathcal{H}}=\left(\begin{array}{cc}
c & 0 \\
0 & -c
\end{array}\right) \otimes\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)+\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \otimes\left(\begin{array}{cc}
c & 0 \\
0 & -c
\end{array}\right) \\
& \Rightarrow \mathcal{H}_{0}=\left(\begin{array}{cccc}
2 c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 c
\end{array}\right)
\end{aligned}
$$

(4) The perturbation matrix is easily obtained by noticing that $V$ operates only on the last component of the state vectors:

$$
\begin{aligned}
& V|S S\rangle=V|A A\rangle=\frac{u}{2}\left(\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right) \quad V|S A\rangle=V|A S\rangle=\frac{u}{2}\left(\begin{array}{c}
0 \\
0 \\
0 \\
-1
\end{array}\right) \\
& \Rightarrow V=\frac{u}{4}\left(\begin{array}{cccc}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{array}\right)
\end{aligned}
$$

(5) We can see that the central block of the hamiltonian is degenerate, and the perturbation couples these degenerate states, so before using perturbation theory we must remove the degeneracy:

$$
\mathcal{H}^{(\operatorname{deg})}=\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)+\frac{u}{4}\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right)
$$

After diagonalization, the central block is:

$$
\mathcal{H}^{(d e g)}=\frac{u}{4}\left(\begin{array}{ll}
2 & 0 \\
0 & 0
\end{array}\right)
$$

Where we used the symmetric and antisymmetric states. We are only interested in the diagonal of the hamiltonian for the first order perturbation, meaning:

$$
E_{n}=E_{n}^{(0)}+V_{n, n}+\mathcal{O}\left(u^{2}\right)
$$

So we don't have to worry about off-diagonal values that were changed by this diagonalization. The energy values are (from the ground state up):

$$
E_{0} \approx-2 c+\frac{u}{4}
$$

$$
\begin{aligned}
& E_{1} \approx 0 \\
& E_{2} \approx \frac{u}{2} \\
& E_{3} \approx 2 c+\frac{u}{4}
\end{aligned}
$$

(6) The ground state energy up to the second order is of the form:

$$
\begin{aligned}
& E_{0}=E_{n}^{(0)}+V_{0,0}+\sum_{m \neq 0} \frac{\left|V_{m, 0}\right|^{2}}{E_{0}^{(0)}-E_{m}^{(0)}}+\mathcal{O}\left(u^{3}\right) \\
& E_{0}^{(2)}=\frac{u^{2}}{16}\left(\frac{1}{-2 c}+\frac{1}{-2 c}+\frac{1}{-4 c}\right)=-\frac{5 u^{2}}{64 c}
\end{aligned}
$$

So the ground state energy is:

$$
E_{0} \approx-2 c+\frac{u}{4}-\frac{5 u^{2}}{64 c}
$$

