E1470: Current in a three site system

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The problem:

A particle in a three site system is described using the standard basis: $|x=1\rangle, |x=2\rangle, |x=3\rangle$, with the Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & c & 0 \\ c & 0 & c \\ 0 & c & 0 \end{pmatrix}$$

(1) Find the eigenvalues of the Hamiltonian: E_0, E_+, E_-.
(2) Find the eigenstates of the Hamiltonian: $|A\rangle, |S_+\rangle, |S_-\rangle$.

The particle has a charge e.

(3) Find the matrix representation of the current operator I for current from site 1 to 2.
(4) Find the observable values of the current operator you defined.

The particle has been prepared in the state $|x=1\rangle$ at $t=0$.

(5) Find the probability of measuring $I=0$ at time t.

The solution:

(1) Using diagonalization, we solve the equation,

$$det(\mathcal{H} - \lambda\mathbb{I}) = 0$$

thus finding the eigenvalues:

$$E_0 = 0 \qquad E_\pm = \pm\sqrt{2}c$$

(2) The eigenstates of the Hamiltonian are:

$$|A\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)$$

$$|S_+\rangle = \frac{1}{2}(|1\rangle + \sqrt{2}|2\rangle + |3\rangle)$$

$$|S_-\rangle = \frac{1}{2}(|1\rangle - \sqrt{2}|2\rangle + |3\rangle)$$

(3) We find the current operator in two ways:

1. Using the projection operator - the current from site 1 to site 2 being the rate of change of the projection operator $\hat{P}^1$.

   This can be inferred from the probabilty continuity equation: $\nabla \cdot J + \frac{\partial\rho}{\partial t} = 0$.

$$\hat{I}_{1\to 2} = -e\frac{\partial \hat{P}^1}{\partial t} = -ie\left[\hat{\mathcal{H}}, \hat{P}^1\right] = e\begin{pmatrix} 0 & ic & 0 \\ -ic & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2. Using spatial phase. We write the hamiltonian:

$$\mathcal{H} = \begin{pmatrix} 0 & ce^{i\phi} & 0 \\ ce^{-i\phi} & 0 & c \\ 0 & c & 0 \end{pmatrix}$$

The current is defined as $\hat{I} = -e\frac{\partial\hat{\mathcal{H}}}{\partial\phi}|_{\phi=0}$,

$$\hat{I} = -ie \begin{pmatrix} 0 & ce^{i\phi} & 0 \\ -ce^{-i\phi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} |_{\phi=0} = -e \begin{pmatrix} 0 & ic & 0 \\ -ic & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4) Finding the observable values using diagonalization for the 2×2 block:

$$I_0 = 0 \qquad I_\pm = \pm ce$$

(5) Using the energy basis, one can describe the wave function at t=0, and the current eigenstate corresponding to zero current.

$$|\psi(t=0)\rangle = |1\rangle = \frac{1}{\sqrt{2}}|A\rangle + \frac{1}{2}|S_+\rangle + \frac{1}{2}|S_-\rangle$$

$$|I=0\rangle = -\frac{1}{\sqrt{2}}|A\rangle + \frac{1}{2}|S_+\rangle + \frac{1}{2}|S_-\rangle$$

$$|\langle\psi(t)|I=0\rangle|^2 = \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}\mathrm{e}^{-i\sqrt{2}ct} & \frac{1}{2}\mathrm{e}^{i\sqrt{2}ct} \end{pmatrix} \cdot \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \sin^4\left(\frac{ct}{\sqrt{2}}\right)$$