E143: 2 sites Bose-Einstein Condensation experiment

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The solution:

(1) The wave function that describes the particle is the symetric eigenstate:

$$\psi(x) = e^{(-\frac{(x+a)^2}{2\sigma^2})} + e^{(-\frac{(x-a)^2}{2\sigma^2})}$$

In order to find the momentum distribution function, we will have to perform a Fourier Transform. We know that for a Gaussian function, the F.T. is:

$$F.T.\left[e^{-\frac{x^2}{2\sigma^2}}\right] \propto e^{-\frac{a^2k^2}{2}}$$

Therefore:

$$\tilde{\psi}(k) = F.T[\psi(x)] = e^{-ika} \cdot e^{-\frac{\sigma^2k^2}{2}} + e^{+ika} \cdot e^{-\frac{\sigma^2k^2}{2}} = e^{-\frac{\sigma^2k^2}{2}} \cdot 2cos(ka)$$

For the symetric wave function, the distribution function will be:

$$P(k) = \left|\tilde{\psi}(k)\right|^2 = e^{-\sigma^2k^2} \cdot 4cos^2(ka)$$

(2) The wave function describes the summation of the
RIGHT location $|R >= e^{(-\frac{(x-a)^2}{2\sigma^2})}$ and the LEFT location $|L >= e^{(-\frac{(x+a)^2}{2\sigma^2})}$,
therefore the symetric wave function can be written as $\psi(x) \equiv |S >= |L > +|R >$.
It is easy to see that the anti-symetric wave function, $|A >$ will be described as
$|A >= |L > -|R >= e^{(-\frac{(x+a)^2}{2\sigma^2})} - e^{(-\frac{(x-a)^2}{2\sigma^2})}$
Hance, after a Fourier Transform, we will get:

$$|\tilde{A}(k) >= F.T[|A >] = e^{-\frac{\sigma^2k^2}{2}} \cdot 2i \cdot sin(ka)$$

And for the anti-symetric wave function, the distribution function will be:

$$P(k) = e^{-\sigma^2k^2} \cdot 4sin^2(ka)$$

(3) For a 2-sites system with a potential u and transmission amplitude c we will get the Hemiltonian:

$$\mathcal{H} = \begin{pmatrix} u/2 & c \\ c & -u/2 \end{pmatrix}$$

The first scenario describes "sudden change" where a potential u is turned on. The condition $u >> c$ allows us to rewrite the Hemiltonian as :

$$\mathcal{H} = \begin{pmatrix} u/2 & 0 \\ 0 & -u/2 \end{pmatrix}$$

The energies are $\pm u/2$ and therefore we can write the time dependant wave function:

$$\psi_t(x) = e^{-i\frac{u}{2}t}|L > +e^{+i\frac{u}{2}t}|R >$$

After a Fourier Transform we will get:

$$\tilde{\psi}_t(k) = [e^{-i(ka+\frac{u}{2}t)} + e^{+i(ka+\frac{u}{2}t)}] \cdot e^{-\frac{\sigma^2 k^2}{2}} = cos(ka + \frac{u}{2}t) \cdot e^{-\frac{\sigma^2 k^2}{2}}$$

Therefore the momentum distribution function will be:

$$P(k) = cos^2(ka + \frac{u}{2}t) \cdot e^{-\sigma^2 k^2}$$

(4) In order to find the time t where minimum amount of particles will stay in the system (where most of the particles will be in the anti-symetric system) we should remember that the anti-symetric momentum distribution function is proportional to $sin^2(ka)$ and the time t will be the time where the function $P(k)$ changes from $cos^2(ka)$ to $sin^2(ka)$.
The condition is:

$$\frac{ut}{2} = \frac{\pi}{2} + 2\pi n \quad \longmapsto \quad t = \frac{\pi}{u} + \frac{4\pi n}{u}$$

(5) Scenario 2 describes an Adiabatic change where the potential u is slowly turned on, and when it is much larger than the transmission amplitude c, $u >> c$, it is suddenly turned off. At that point $u = 0$ and after time t the momentum distribution is examined.
We will assume that the potential is set in a way where the left state $|L>$ has higher energy level than the right state $|R>$. We Know that in 2 state system, the symetric state is the Base state, therefore the lowest state.
During an adiabatic change, the wavefunction might change, but the energy levels don't! So in time $t = 0$ (the first moment when $u = 0$) the wave function is $\psi = |R>$.
The Hemiltonian based on the standard eigen states will be:

$$\mathcal{H} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$$

The Hemiltonian based on the symetric and anti-symetric states will be:

$$\mathcal{H} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

Now we have to take the wavefunction and find it's time evolution:

$$\psi_t(x) = \hat{U}(t)|R> = |S> \cdot e^{-ict} - |A> \cdot e^{+ict}$$

$$\tilde{\psi}(k) = e^{-ict} \cdot e^{-\frac{\sigma^2 k^2}{2}} \cdot 2cos(ka) - e^{+ict} \cdot e^{-\frac{\sigma^2 k^2}{2}} \cdot 2i \cdot sin(ka)$$

$$P(k) = \left|\tilde{\psi}(k)\right|^2 = 4e^{-\sigma^2 k^2} \cdot [1 + sin(2ka) \cdot sin(2ct)]$$

(6) The wave function is occilating between the two optional states $|L>$, $|R>$.
The time t when the wave function is a perfect Gaussian (and does not modulate) is the time when the wave function is exactly in one of the two states.
After a full time period the wavefunction returns to it's original location ($|R>$). after half a period the function will be in the other state ($|L>$). therefore......

$$t = \frac{\pi}{\omega} \cdot integer \quad \rightarrow \quad \omega = 2c \quad \rightarrow \quad t = \frac{\pi}{2c} \cdot integer$$

`The problem???`