Ex1400: Two Particles in Two Sites, Oscillations

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The Problem:

Let us consider a system of two sites $1,2$ with two particles a,b. The system is symmetric under reflection, and the hopping amplitude per unit time is Δ (suppose real, without loss of generality). The standard basis is:

$$|x_a, x_b>= (|1,1>, |1,2>, |2,1>, |2,2>)$$

1. Write the Hamiltonian in the standard basis. Notice that in an infinitismal time step at the most one particle can hop from one site to another. The probability of simultaneous passage of both particles is zero.
2. Write the matrix representation of the transposition operator $\hat{T}$ which swaps the particles.
3. Define a set of states $|i = L, R, S, A>$ where $\hat{T}$ representation is diagonal. Tip: these states are "left", "right", "symmetric" and "anti-symmetric".
From now on suppose the particles are the same spinless-bosons.
4. Write the Hamiltonian in the basis $|i = L, R, S>$.
5. In the same basis write the reflection operator $\hat{R}$.
6. Define a set of states in which the reflection operator representation is diagonal.
7. Find the eigenstates of the Hamiltonian, write them in the standard basis.
8. What are the eigen-energies?
Suppose the system is prepared in the symmetric state $|S>$ at time t=0.
9. One measures the number of particles in the left site. What are the possible options?
10. What is the time period of the system oscillations?
11. What is the answer for (9) after a half time period?

Solution:
1. Since there is no simultaneous approach of both particles, there is no coupling between $(|1,1>, |2,2>); (|1,2>, |2,1>)$. Therefore the Hamiltonian representation in the standard basis is:

$$\hat{H} \mapsto \begin{pmatrix} 0 & \Delta & \Delta & 0 \\ \Delta & 0 & 0 & \Delta \\ \Delta & 0 & 0 & \Delta \\ 0 & \Delta & \Delta & 0 \end{pmatrix}$$

2. By definition,

$$\hat{T}|1,1>= |1,1>; \hat{T}|1,2>= |2,1>; \hat{T}|2,1>= |1,2>; \hat{T}|2,2>= |2,2>$$

$$\hat{T} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. We notice σ_x at the middle, hence the eigenstates of the transposition operator are:

$$|S>= \frac{1}{\sqrt{2}}(|1,2> + |2,1>); |A>= \frac{1}{\sqrt{2}}(|1,2> - |2,1>) \tag{1}$$

$$|L>= |1,1>; |R>= |2,2>; \tag{2}$$

4. Since the particles are the same spinless-bosons, the above anti-symmetric state $|A>$ is not relevant. One can calculate:

$$\hat{H}|L>=\sqrt{2}\Delta|S>; \hat{H}|R>=\sqrt{2}\Delta|S>$$

$$\hat{H}|S>=\sqrt{2}\Delta(|L>+|R>); \hat{H}|A>=0$$

Hence, the representation of the Hamiltonian in the L,R,S basis is:

$$\hat{H}\mapsto\sqrt{2}\Delta\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$

5. The reflection operator maps $|L>$ to $|R>$ and vise versa, and do not change the symmetric state $|S>$. Therefore,

$$\hat{R}\mapsto\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$$

6. It is easy to see that $|S>$ is one of the eigenstates of the reflection operator. The other eigenstates are:

$$|+>=\frac{1}{\sqrt{2}}(|L>+|R>); |->=\frac{1}{\sqrt{2}}(|L>-|R>) \tag{3}$$

7. We operate the Hamiltonian on the above states:

$$\hat{H}|S>=2\Delta|+>; \hat{H}|+>=2\Delta|S>; \hat{H}|->=0$$

Hence we have the representation of $\hat{H}$ in $|i=S,+,->$ basis:

$$\hat{H}\mapsto 2\Delta\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$$

We notice Pauli matrix σ_x, with eigenstates:

$$|E_+>=\frac{1}{\sqrt{2}}(|S>+|+>); E_+=2\Delta$$

$$|E_->=\frac{1}{\sqrt{2}}(|S>-|+>); E_-=-2\Delta$$

$$|E_0>=|->; E_0=0$$

Now we shall use (1),(2),(3) and get:

$$|E_+>=\frac{1}{\sqrt{2}}|1,2>+\frac{1}{2}(|1,1>+|2,2>) \tag{4}$$

$$|E_->=\frac{1}{\sqrt{2}}|1,2>-\frac{1}{2}(|1,1>+|2,2>) \tag{5}$$

$$|E_0>=|->=\frac{1}{\sqrt{2}}(|1,1>-|2,2>) \tag{6}$$

8. Knowing the eigenvalues of σ_x: ± 1, we have:

$$\hat{H} \mapsto 2\Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

9. In the symmetric state there is one particle in the left site and one particle in the right site. Therefore <u>1</u> particle will be observed anyway at the left site.
10.

$$|S>= \frac{1}{\sqrt{2}}(|E_+> + |E_->)$$

The frequency is given by the difference between energies which is 4Δ. Hence,

$$T = \frac{2\pi}{4\Delta} = \frac{\pi}{2\Delta}$$

11. Therefore the state at time $\frac{T}{2}$ is given by:

$$|\psi(\frac{T}{2})>= \frac{1}{\sqrt{2}}(|E_+> e^{-i2\Delta\frac{T}{2}} + |E_-> e^{i2\Delta\frac{T}{2}})$$

Substituting (4),(5),(6) in the last expression one can have:

$$|\psi(\frac{T}{2})>= -\frac{i}{\sqrt{2}}(|1,1> + |2,2>)$$

So both particles are in left site or in the right site (with probability of half-half). One deduces that <u>0 or 2</u> are the possible options for the number of the particles in the left site.