E139: A partical in a four site system

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The problem

A system with four sites described by the Hamiltonian:

$$H = \begin{pmatrix} 0 & 1 & 0 & c \\ 1 & 0 & c & 0 \\ 0 & c & 0 & 1 \\ c & 0 & 1 & 0 \end{pmatrix}$$

The electrical charge is e.

(1) Write the Hamiltonian if we add a magnetic flux Φ through the ring that connected site $|1\rangle$ to site $|2\rangle$.
(2) Define the current operator using the formula $I = -\frac{dH}{d\Phi}_{\Phi=0}$. We assume that the flux is zero.
(3) Write in the standart basis the eigenstates and the energies eigenvalues when $c = 0$. Use in $|S_1\rangle, |A_1\rangle, |S_2\rangle, |A_2\rangle$.
(4) Write the Hamiltonian in the new basis.
(5) Write the energies eigenvalues (E_1, E_2, E_3, E_4) and the eigenstates $|E_1\rangle, |E_2\rangle, |E_3\rangle, |E_4\rangle$ of the system as a superposition of the states $|S_1\rangle, |A_1\rangle, |S_2\rangle, |A_2\rangle$.
(6) Write the energies eigenstats aforementioned in the standart basis.

The partical is plased in site one $|\Psi(t=0)\rangle = |1\rangle$.

(7) Calculate the current that flow through the ring as a time function $\langle \Psi(t) | I | \Psi(t) \rangle$.

The solution

(1) Magnetic flux add a phase to the Hamiltonian.

$$H = \begin{pmatrix} 0 & e^{-i\varphi} & 0 & c \\ e^{i\varphi} & 0 & c & 0 \\ 0 & c & 0 & 1 \\ c & 0 & 1 & 0 \end{pmatrix}$$

$$\varphi = e \cdot \Phi$$

(2) The current operator :

$$I = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) For $c = 0$. The Hamiltonian is :

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The eigenstates of the Hamiltonian are the states which are symmetrical or antisymmetrical .

$$|S_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$|A_1\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

$$|S_2\rangle = \frac{1}{\sqrt{2}}(|3\rangle + |4\rangle)$$

$$|A_2\rangle = \frac{1}{\sqrt{2}}(|3\rangle - |4\rangle)$$

The energies eigenvalues :

$$E_1 = 1 \; ; E_2 = 1 \; ; E_3 = -1 \; ; E_4 = -1$$

(4) In order to fined the Hamiltonian in the new basis, we need to find a matrix that takes us from the old basis to the new one.

The new basis is : $\{|S_1\rangle,|S_2\rangle,|A_1\rangle,|A_2\rangle\}$

The transformation matrix is :

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$H_{new} = T^{-1}H_{old}T = \begin{pmatrix} 1 & c & 0 & 0 \\ c & 1 & 0 & 0 \\ 0 & 0 & -1 & -c \\ 0 & 0 & -c & -1 \end{pmatrix}$$

(5) The energies eigenstates of the system is :

$$|E_1\rangle = \frac{1}{\sqrt{2}}\left(|S_1\rangle + |S_2\rangle\right)$$

$$|E_2\rangle = \frac{1}{\sqrt{2}}\left(|S_1\rangle - |S_2\rangle\right)$$

$$|E_3\rangle = \frac{1}{\sqrt{2}}\left(|A_1\rangle + |A_2\rangle\right)$$

$$|E_4\rangle = \frac{1}{\sqrt{2}}\left(|A_1\rangle - |A_2\rangle\right)$$

The energies eigenvalues :

$$E_1 = 1 + c \; ; E_2 = 1 - c \; ; E_3 = -1 - c \; ; E_4 = -1 + c$$

(6) The energies eigenstates of the system in the standart basis is :

$$|E_1\rangle = \frac{1}{2}\left(|1\rangle + |2\rangle + |3\rangle + |4\rangle\right)$$

$$|E_2\rangle = \frac{1}{2}\left(|1\rangle + |2\rangle - |3\rangle - |4\rangle\right)$$

$$|E_3\rangle = \frac{1}{2}\left(|1\rangle - |2\rangle + |3\rangle - |4\rangle\right)$$

$$|E_4\rangle = \frac{1}{2}\left(|1\rangle - |2\rangle - |3\rangle + |4\rangle\right)$$

(7)

$$|\Psi(t=0)\rangle = |1\rangle = \frac{1}{2}\left(|E_1\rangle + |E_2\rangle + |E_3\rangle + |E_4\rangle\right)$$

$$|\Psi(t)\rangle = \frac{1}{2}\left(e^{-iE_1t}|E_1\rangle + e^{-iE_2t}|E_2\rangle + e^{-iE_3t}|E_3\rangle + e^{-iE_4t}|E_4\rangle\right)$$

$$A = \langle 1|\Psi(t)\rangle = \frac{1}{4}\left(e^{-iE_1t} + e^{-iE_2t} + e^{-iE_3t} + e^{-iE_4t}\right) = cos(t)cos(ct)$$

$$B = \langle 2|\Psi(t)\rangle = \frac{1}{4}\left(e^{-iE_1t} + e^{-iE_2t} - e^{-iE_3t} - e^{-iE_4t}\right) = -isin(t)cos(ct)$$

$$\langle\Psi(t)\left|I\right|\Psi(t)\rangle = i(A^*B - B^*A) = 2Im[B^*A] = 2Im[\langle 2\left|\Psi(t)\right\rangle^* \langle 1\left|\Psi(t)\right\rangle]$$

The current that flow through the ring as a time function is :

$$I_{(t)} = e \cdot sin(2t)cos^2(ct)$$