# E138: system with three sites

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**The problem:**

A system with three sites described by the hamiltonian

$$\hat{\mathcal{H}} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

and the position matrix:

$$\hat{\mathcal{X}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

. The purpose of this problem is to find the probability $P(t)$ to find the particle at site left after time $t$ and the current as a function of $t$.

(1) Write the Hamiltonian if we add a magnetic flux $\Phi$ through the ring that connects the right site to the left.
(2) Define the current operator using the formula $I = -\frac{dH}{d\phi}$.
(3) Write in the standard basis the eigenstates and the energies eigenvalues.note that the hamiltonian is symetrical
(4) Calculate the probability $P(t)$ to find the particle in the left site ufter a while .
(5) Calculate the current $\langle I \rangle_t$ that flow through the ring.

**The solution:**

(1) The Hamiltonian is

$$\hat{\mathcal{H}} = \begin{pmatrix} 2 & -1 & -\mathrm{e}^{-i\phi} \\ -1 & 2 & -1 \\ -\mathrm{e}^{i\phi} & -1 & 2 \end{pmatrix}$$

and the phase is $\phi = e\Phi$

(2) The current when $\Phi = 0$ is:

$$I = -\frac{dH}{d\Phi} = e \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

(3)The momentum eigenvalues are: $k_0 = 0$ , $k_+ = \frac{2\pi}{3}$ , $k_- = -\frac{2\pi}{3}$ and the eigenstates are (because the hamiltonian is symmetrical to translations we have $S$ and $antiS$ solutions):

$$|k_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$$

$$|k_+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \mathrm{e}^{i\frac{2\pi}{3}}|1\rangle + \mathrm{e}^{-i\frac{2\pi}{3}}|2\rangle)$$

$$|k_-\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \mathrm{e}^{-i\frac{2\pi}{3}}|1\rangle + \mathrm{e}^{i\frac{2\pi}{3}}|2\rangle)$$

and the energies are:

$$E_0 = 0$$

$$E_+ = 3$$

$$E_- = 3$$

(4) The probability to find the particle at site left after time $t$ is:

$$\psi(t) = \frac{1}{\sqrt{3}}(|k_0\rangle + \mathrm{e}^{-i\omega t}|k_+\rangle + \mathrm{e}^{-i\omega t}|k_-\rangle)$$

where $\omega = E_+ - E_0 = 3$.
With some algebra we get :

$$P(t) = |\langle 0|\psi(t)\rangle|^2 = |\frac{1}{\sqrt{3}}(\langle 0|k_0\rangle + \mathrm{e}^{-i\omega t}\langle 0|k_+\rangle + \mathrm{e}^{-i\omega t}\langle 0|k_-\rangle)|^2 = |\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{3}} + \mathrm{e}^{-i\omega t}\frac{1}{\sqrt{3}} + \mathrm{e}^{-i\omega t}\frac{1}{\sqrt{3}})|^2$$

$= \frac{1}{9}(5 + 4\cos(\omega t))$

(5) Using $\psi(t)$ from (4) and the definition of $I$ from (2) we get the current:

$$\psi_0 = \langle 0|\psi(t)\rangle = \frac{1}{3}(1 + \mathrm{e}^{-i\omega t} + \mathrm{e}^{-i\omega t})$$

$$\psi_2 = \langle 2|\psi(t)\rangle = \frac{1}{3}(1 + \mathrm{e}^{-i\omega t - i\frac{2\pi}{3}} + \mathrm{e}^{-i\omega t + i\frac{2\pi}{3}})$$

$$\langle I\rangle_t = \langle\psi(t)|I|\psi(t)\rangle = e(-i\psi_0 * \psi_2 + i\psi_2 * \psi_0) = \frac{2}{3}e\sin\omega t$$