E136: A three site system

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The problem:

A three site system is descibed with the standard base $|1\rangle, |0\rangle, |{-1}\rangle$ through the hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & c & 1 \\ c & u & c \\ 1 & c & 0 \end{pmatrix}$$

assume that the site in the center $|0\rangle$ is disconnected ($c = 0$).
(1) Using the standard base write the eigenstates $|S\rangle, |S_0\rangle, |A\rangle$ and their eigenvalues.
(2) Write the hamiltonian using the new base that you have found.
from now on assume that $u = 1$ and $c \neq 0$.
(3) Write the eigenvalues E_1, E_2, E_3, and the eigenstates $|E_1\rangle, |E_2\rangle, |E_3\rangle$ of the system as a superposition of the states $|S\rangle, |S_0\rangle, |A\rangle$.
(4) Write the new eigenvalues using the standard base.
(5) What is the probobilty to find the partice in the state $|0\rangle$ after a time interval t if it is prepered at time $t = 0$ in the state $|-1\rangle$.
(6) What is the probobilty to find the partice in the state $|0\rangle$ after a time interval t if it is prepered at time $t = 0$ in the state $|A\rangle$.

The solution:

(1) The eigenstates $|S\rangle, |S_0\rangle, |A\rangle$ are:

$$|S\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$|S_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|A\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The eigenvalues are:

$$\mathcal{H}|S\rangle = |S$$

$$\mathcal{H}|S_0\rangle = u|S_0\rangle$$

$$\mathcal{H}|A\rangle = -|A$$

(2) The new hamiltonian is:

$$\widetilde{\mathcal{H}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(3) Let us operate the hamiltonian on the states $|S\rangle\,,|S_0\rangle\,,|A\rangle$:

$$\mathcal{H}\,|S\rangle = |S\rangle + \sqrt{2}c|S_0\rangle$$

$$\mathcal{H}\,|S_0\rangle = \sqrt{2}c|S\rangle + |S_0\rangle$$

$$\mathcal{H}\,|A\rangle = -|A\rangle$$

Now we can write the hamiltonian using the base $|S\rangle\,,|S_0\rangle\,,|A\rangle$ as a block diagonal matrix:

$$\widetilde{\mathcal{H}} = \begin{pmatrix} 1 & \sqrt{2}c & 0 \\ \sqrt{2}c & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenstates $|E_1\rangle\,,|E_2\rangle\,,|E_3\rangle$ are:

$$|E_1\rangle = \frac{1}{\sqrt{2}}(|S\rangle + |S_0\rangle)$$

$$|E_2\rangle = \frac{1}{\sqrt{2}}(|S\rangle - |S_0\rangle)$$

$$|E_3\rangle = |A\rangle$$

The eigenvalues are:

$$\widetilde{\mathcal{H}}|E_1\rangle = (1+\sqrt{2}c)|E_1\rangle$$

$$\widetilde{\mathcal{H}}|E_2\rangle = (1-\sqrt{2}c)|E_2\rangle$$

$$\widetilde{\mathcal{H}}|E_3\rangle = -|E_3\rangle$$

(4) The eigenstates $|E_1\rangle\,,|E_2\rangle\,,|E_3\rangle$ writen in the standard base are:

$$|E_1\rangle = \frac{1}{2}(|1\rangle + \sqrt{2}|0\rangle + |-1\rangle)$$

$$|E_2\rangle = \frac{1}{2}(|1\rangle - \sqrt{2}|0\rangle + |-1\rangle)$$

$$|E_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$$

(5) Let us write the standard base using $|E_1\rangle\,,|E_2\rangle\,,|E_3\rangle$

$$|1\rangle = \frac{1}{2}(|E_1\rangle + |E_2\rangle + \sqrt{2}|E_3\rangle)$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle)$$

$$|-1\rangle = \frac{1}{2}(|E_1\rangle + |E_2\rangle - \sqrt{2}|E_3\rangle)$$

We know that at $t = 0$ the state of the particle is:

$$\Psi_{(t=0)} = |-1\rangle$$

Then the state of the particle, after time t will be:

$$\Psi_{(t)} = \frac{1}{2}(e^{-(iE_1 t)}|E_1\rangle + e^{-(iE_2 t)}|E_2\rangle - \sqrt{2}e^{-(iE_3 t)}|E_3\rangle) = \frac{1}{2}(e^{-i(1+\sqrt{2}c)t}|E_1\rangle + e^{-i(1-\sqrt{2}c)t}|E_2\rangle - \sqrt{2}e^{it}2|E_3\rangle)$$

The probobilty of finding the particle in state $|0\rangle$ after a time interval t is:

$$|\langle x = 0|\Psi_{(t)}\rangle|^2 = |\frac{1}{2^{\frac{3}{2}}}(e^{-i(1+\sqrt{2}c)t} - e^{-i(1-\sqrt{2}c)t})|^2 = \frac{1}{2}\sin^2(\sqrt{2}ct)$$

(6) The state $|A\rangle$ is an eigenstate of the hamiltonian, and therefore it is a stationary stae which we know is a superposition of states $|1\rangle$ and $|-1\rangle$. Therefore the probobilty of finding the particle at state $|0\rangle$ is zero.