E1344: Oscillations between a site and a ring

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The problem:

A given ring with length $L = 1$ has N sites, which have equal potential ($V = 0$). The hopping amplitude of a particle per time unit between neighbouring sites is c. Another site is added at the center of the ring. The relation energy of the particle in the central site is ε_0. The hopping amplitude per time unit from the central site to any of the other sites along the ring is c_0. The system is then placed in a magnetic field, so that the total flux through the ring is Φ. The charge of the particle is e. From the following analysis, it is derived that the particle will perform oscillations between the site and the ring. This is a generalization of oscillations in a system of two sites.

(1) Write the Hamiltonian $H_{ring}(p)$ of a particle in the ring described above (without the central site).

(2) Write the eigenenergies E_n of the particle in the ring (without the central site).

(3) Calculate the coupling $\langle E_n|H|\varepsilon_0\rangle$ of the central site to the states of the ring.

(4) What is the oscillation frequency Ω of the particle?

(5) For a given magnetic flux - What should be the coupled energy ε_0 in order to get full oscillations?

(6) For oscillation $\Omega(\Phi)$ there is periodical dependency in the magnetic flux. What is the period, based on Aharonov-Bohm?

Answer the questions, using given data only.

The solution:

(1) From the lecture notes [9.1], we know that:

$$\widehat{H} = ce^{-ia(\widehat{p}-A)} + ce^{ia(\widehat{p}-A)} + const.$$

Because of gauge consideration, we can determine that $const = 0$.

In addition, we know that $a = \dfrac{L}{N} = \dfrac{1}{N}$, and $A = \dfrac{\mathrm{e}\Phi}{L}$. Therefore:

$$\widehat{H} = ce^{-i\frac{1}{N}(\widehat{p}-\mathrm{e}\Phi)} + ce^{i\frac{1}{N}(\widehat{p}-\mathrm{e}\Phi)} = 2c\cdot\cos\left(\frac{1}{N}(\widehat{p}-\mathrm{e}\Phi)\right)$$

(2) Considering the hamiltonian, the eigenstates are the momentum states and the eigenvalues are as follows:

$$E_n = 2c\cdot\cos\left(\frac{1}{N}(2\pi n-\mathrm{e}\Phi)\right)$$

(3) Due to the fact that $\widehat{H} = \widehat{H}(\hat{p})$, the eigenstates $|E_n\rangle = |k_n\rangle$. Hence,

$$\begin{aligned}
\langle E_n|H|\varepsilon_0\rangle &= \langle k_n|H|\varepsilon_0\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=1}^{N} e^{ik_n x}\langle x|H|\varepsilon_0\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=1}^{N} e^{ik_n x}\cdot c_0 \\
&= \frac{1}{\sqrt{N}} N\delta_{n,0} c_0 \\
&= \sqrt{N} c_0 \delta_{n,0}
\end{aligned}$$

(4) Due to the coupling, there will be oscillations between the central site to the ring with 0 momentum:

$$\left(\begin{array}{cc|cccc}
\varepsilon_0 & c_0\sqrt{N} & & & & \\
c_0\sqrt{N} & E_0 & & & & \\
\hline
 & & E_1 & & & \\
 & & & E_2 & & \\
 & & & & \ddots & \\
 & & & & & E_n
\end{array}\right)$$

Because of gauge consideration, we can determine for the 2×2 subspace that:

$$\begin{pmatrix} \frac{\varepsilon_0 - E_0}{2} & c_0\sqrt{N} \\ c_0\sqrt{N} & -\frac{\varepsilon_0 - E_0}{2} \end{pmatrix}$$

From the lecture notes [36.1], we learn that the evolution of any system whose states form a dim=2 Hilbert space can always be described by using a precession picture. Therefore, the subspace can be rewritten as:

$$\frac{\varepsilon_0 - E_0}{2}\cdot\sigma_z + c_0\sqrt{N}\cdot\sigma_x = \vec{\Omega}\cdot\vec{S}$$

Where $\vec{\Omega} = \left(2c_0\sqrt{N}, 0, \varepsilon_0 - E_0\right)$, and $E_0 = 2c\cdot\cos\left(\frac{e\Phi}{N}\right)$.
Therefore:

$$\Omega = \sqrt{\left(\varepsilon_0 - 2c\cdot\cos\left(\frac{e\Phi}{N}\right)\right)^2 + 4c_0^2 N}$$

(5) Using the Rabi formula:

$$P(t) = 1 - \sin^2(\theta_0)\sin^2\left(\frac{\Omega t}{2}\right)$$

Where θ_0 is the angle between the $\widehat{z}$ axis and the rotation axis ($\vec{\Omega}$).
The condition to get a full oscillation is $\theta_0 = \dfrac{\pi}{2}$, and in our case θ_0 is:

$$\theta_0 = \arctan\left(\frac{2c_0\sqrt{N}}{\varepsilon_0 - 2c\cdot\cos\left(\frac{\mathrm{e}\Phi}{N}\right)}\right)$$

Therefore:

$$\varepsilon_0 = 2c\cdot\cos\left(\frac{\mathrm{e}\Phi}{N}\right)$$

(6) The oscillation $\Omega(\Phi)$ has a period depending on the the cosine function, as we can see in the equation from section (4).
Therefore, the necessary condition is:

$$\frac{\mathrm{e}\Phi}{N} = 2\pi k\,, \quad k\in\mathbb{Z} \quad\Longrightarrow\quad \Phi = \frac{2\pi N}{\mathrm{e}}k$$

and the condition is:

$$\Phi \longmapsto \Phi + \frac{2\pi N}{\mathrm{e}}k$$