Ex1328: Transmission Through a Junction in a One-Dimensional Lattice

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The question:

In this problem it is required to find the transmission coefficient $g = |t|^2$ of a particle of momentum k through a junction in a one-dimensional lattice.
The given parameters are: Lattice constant a, "hopping" frequency between nearest neighbors c_0 and "hopping" frequency between the $n = \pm 1$ sites c_J.

(1) In this section we assume an unperturbated lattice, where instead of a junction we have another site $n = 0$. Write down the unperturbated Hamiltonian H_0 and the velocity operator v as a function of the momentum operator p.
(2) Write down the matrix $H_{n,m}$ representing the perturbated Hamiltonian. It is sufficient to write the $4x4$ sub-matrix for $n, m = -2, -1, 1, 2$.
(3) Write a system of two equations for the transmission and reflection amplitudes (r, t) of the junction.
(4) Write an expression for the transmission coefficient g as a function of k.
(5) For a very large of very small $g \propto |v_k|^2$, write down the proportion coefficients.

The solution:

(1)

$$H_0 = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & 0 & c_0 & 0 & \\ \ddots & c_0 & 0 & c_0 & \ddots \\ & 0 & c_0 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

We notice that:
$H_0 = c_0 \left(D + D^\dagger\right) = 2c_0 \left(\frac{e^{-ipa} + e^{+ipa}}{2}\right) = 2c_0 \cos(ap)$

To find v:
$v = \frac{dx}{dt} = i[H_0, x] = -i[x, 2c_0 cos(ap)] = -2c_0 i(-a \cdot sin(ap))[x, p] = -2c_0 a \cdot sin(ap)$

(2)

$$H = \begin{pmatrix} 0 & c_0 & 0 & 0 \\ c_0 & 0 & c_J & 0 \\ 0 & c_J & 0 & c_0 \\ 0 & 0 & c_0 & 0 \end{pmatrix}$$

(3)

We assume as guided:

$\psi_n^L = e^{ikan} + re^{-ikan}$
$\psi_n^R = te^{ikan}$

From $H\psi = E_k\psi$ we get:

$$\begin{pmatrix} 0 & c_0 & 0 & 0 \\ c_0 & 0 & c_J & 0 \\ 0 & c_J & 0 & c_0 \\ 0 & 0 & c_0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{-2} \\ \psi_{-1} \\ \psi_1 \\ \psi_2 \end{pmatrix} = E_k \begin{pmatrix} \psi_{-2} \\ \psi_{-1} \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

We obtain the set of 2 equations:

$c_0\psi_{-2} + c_J\psi_1 = E_k\psi_{-1}$
$c_J\psi_{-1} + c_0\psi_2 = E_k\psi_1$

(Notice that we cannot use the first and last equations since the matrix is in fact infinite and they are incorrect)

where $E_k = 2c_0cos(ak) = c_0(e^{ika} + e^{-ika})$

(Same as in the unperturbated hamiltonian).

Inserting the assumed wave functions and energy into the equations, we get:

$c_0(e^{-2ika} + re^{2ika}) + c_J(te^{ika}) = c_0(e^{ika} + e^{-ika})(e^{-ika} + re^{ika})$
$c_J(e^{-ika} + re^{ika}) + c_0(te^{2ika}) = c_0(e^{ika} + e^{-ika})(te^{ika})$
$\Downarrow$
$\frac{c_J}{c_0}te^{ika} = 1 + r$
$\frac{c_J}{c_0}(e^{-ika} + re^{ika}) = t$

(4)

Solving for t from the above system of 2 equations (isolating r in both and equating them) we get:

$t = \frac{2ic_0c_Jsin(ka)}{c_J^2e^{2ika} - c_0^2}$
$\Downarrow$
$|t|^2 = \frac{4c_0^2c_J^2sin^2(ka)}{\left|c_J^2e^{2ika} - c_0^2\right|^2} = \frac{c_J^2v_k^2/a^2}{\left|c_J^2e^{2ika} - c_0^2\right|^2}$

(5)

For very large c_J:

$|t|^2 = \frac{1}{c_J^2}\frac{1}{a^2} \cdot v_k^2$

For very small c_J:

$|t|^2 = \frac{c_J^2}{c_0^4}\frac{1}{a^2} \cdot v_k^2$