latexsym amsmath amssymb bm

E114: Representation of Operators

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The problem :

In a certain basis the operators A and B are represented by the square matrix:

$$\mathcal{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} -b & 0 & 0 \\ 0 & 0 & ib \\ 0 & -ib & 0 \end{pmatrix}$$

(1) Does B have eigenvalue degeneracy?

(2) Show that the operators A and B commute.

(3) Find a new basis where the operators A and B are diagonal.

The solution :

(1) The operator B after digaonalization is:

$$\mathcal{B} = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & b \end{pmatrix}$$

That mean $\mathcal{B}$ has second level of degeneration.

(2) To show that the operators A and B commute we need to show that $[\widehat{A},\widehat{B}\,]= \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$ = 0

By calculating simple matrix multipcation we get the zero value.

Therefore the operators commute.

(3) To digonalise first we need to find the eigenvectors.

|1>$\mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

|2>$\mapsto \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$

|3>$\mapsto \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$

Matrix T is Unitary therefor T$^{-1} = T^{\dagger}$

By using the transformation elements:

$\widetilde{B}$ = $T^{\dagger}\ \mathcal{B}\ T$

We diagonalize B in the new basis

While

$$\mathcal{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

and

$$\mathcal{T}^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

The $\mathcal{T}$ Matrix changes $\mathcal{B}$ base to the diagonal one:

$\widetilde{B}$ = $T^{\dagger}\ \mathcal{B}\ T$

And due to the degeneracy in $\mathcal{B}$, a linear combination of the eigenvectors of $\mathcal{B}$ will digonalise $\mathcal{A}$.

However, by applying $T^{\dagger}\ \mathcal{A}\ T$ We see that $\mathcal{A}$ Remains the same in the new basis

$\widetilde{A}$ = $T^{\dagger}\ \mathcal{A}\ T$ = $\mathcal{A}$

Consequently we have demonstrated that $\mathcal{A}$ and $\mathcal{B}$ are both diagonal in the same basis.