E1060: Change of Basis in a Two Dimensional Hilbert Space

Submitted by: B. Kramer

The problem:

Given the standard orthonormal basis $\{|1\rangle, |2\rangle\}$ of a Hilbert space, we define a new basis $\{|u\rangle, |v\rangle\}$ where $|u\rangle = a|1\rangle + b|2\rangle$.

(1) Find the condition on a,b ensuring $|u\rangle$ a norm of 1.
(2) Define the state $|v\rangle$ so that the new basis will be orthonormal.
(3) Represent the projection operators on the states $|u\rangle, |v\rangle$ in the standard basis.
(4) Could there be other answers to parts (2) and (3)?
(5) Write down the transition matrix from the standard basis to the new basis.
(6) Express the state $|\psi\rangle = 3|1\rangle + 7|2\rangle$ in the new basis.

The solution:

(1) The norm of state $|u\rangle$ is:

$$\langle u|u\rangle = (a^*\langle 1| + b^*\langle 2|)(a|1\rangle + b|2\rangle) = |a|^2 + |b|^2$$

Thus we demand

$$|a|^2 + |b|^2 = 1$$

(2) First we define $|v\rangle = \alpha|1\rangle + \beta|2\rangle$, so that:

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\langle u|v\rangle = (a^*\langle 1| + b^*\langle 2|)(\alpha|1\rangle + \beta|2\rangle) = a^*\alpha + b^*\beta = 0$$

Under these conditions, a simple choice of the parameters will be:

$$\alpha = b^* \ ; \ \beta = -a^*$$

(3) The projection operators are:

$$P^u = |u\rangle\langle u| = (a|1\rangle + b|2\rangle)(a^*\langle 1| + b^*\langle 2|) = |a|^2|1\rangle\langle 1| + |b|^2|2\rangle\langle 2| + a^*b|2\rangle\langle 1| + b^*a|1\rangle\langle 2|$$

$$P^v = |v\rangle\langle v| = (b^*|1\rangle - a^*|2\rangle)(b\langle 1| - a\langle 2|) = |b|^2|1\rangle\langle 1| + |a|^2|2\rangle\langle 2| - a^*b|2\rangle\langle 1| - b^*a|1\rangle\langle 2|$$

Or, using matrix representation:

$$P^u \to \begin{pmatrix} |a|^2 & b^*a \\ a^*b & |b|^2 \end{pmatrix} \ ; \ P^v \to \begin{pmatrix} |b|^2 & -b^*a \\ -a^*b & |a|^2 \end{pmatrix}$$

It's easy to see that $P^u + P^v = \hat{\mathbf{1}}$, as expected.

(4) A second look at the conditions in part (2) tells us we could've added an arbitrary phase to $|v\rangle$:

$$|v\rangle \rightarrow e^{i\varphi}|v\rangle$$

This, however, does not change the solution to part (3):

$$(e^{i\varphi}|v\rangle)(e^{-i\varphi}\langle v|) = |v\rangle\langle v|$$

(5) The transition matrix from the new basis to the standard basis is:

$$T = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}$$

It's inverse is the transition matrix from the standard basis to the new basis:

$$T^{-1} = \begin{pmatrix} a^* & b^* \\ b & -a \end{pmatrix}$$

(6) In the standard basis:

$$|\psi\rangle = 3|1\rangle + 7|2\rangle \rightarrow \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Using our result from part (5):

$$T^{-1}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3a^* + 7b^* \\ 3b - 7a \end{pmatrix}$$

Or, in Dirac notation:

$$|\psi\rangle = (3a^* + 7b^*)|u\rangle + (3b - 7a)|v\rangle$$