E106: Base translation

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The problem:

Given two different bases:

$$\{|\eta_{\rm i}\rangle\} \quad \text{and} \quad \{|\zeta_{\rm i}\rangle\}$$

with the relation:

$$|\eta_1\rangle = {\rm a}|\zeta_1\rangle + {\rm b}|\zeta_2\rangle \quad \text{and} \quad |\eta_2\rangle = {\rm a}|\zeta_1\rangle - {\rm b}|\zeta_2\rangle$$

(1) What should be the condition on a and b so that the base will be ortonormaly.

(2) Express this $|{\rm a}_1\rangle = 3|\zeta_1\rangle + 7|\zeta_2\rangle$ relation in the $|\eta_i\rangle$ base.

The solution:

(1)

$$|\eta_1\rangle = {\rm a}|\zeta_1\rangle + {\rm b}|\zeta_2\rangle$$

$$|\eta_2\rangle = {\rm a}|\zeta_1\rangle - {\rm b}|\zeta_2\rangle$$

We will suppose that $|\zeta_1\rangle$ and $|\zeta_2\rangle$ are unit vectors (it's the base) so:

$$|\eta_\alpha\rangle = \sum_j T_{j\alpha}|\zeta_j\rangle$$

means:

$$T_{j,\alpha} = \langle\zeta_j|\eta_\alpha\rangle = \begin{pmatrix} a & a \\ b & -b \end{pmatrix}$$

For vector $\vec{\eta_1}$ and $\vec{\eta_2}$ to be ortonormaly, we should demand that $T^+T = I$.

From this condition we get:

$$|a|^2 = \frac{1}{2}$$

$$|b|^2 = \frac{1}{2}$$

(2) The transformation matrix is:

$$S = T^{-1} = T^\dagger = \begin{pmatrix} a^* & b^* \\ a^* & -b^* \end{pmatrix}$$

The coefficient vector of $|a_1\rangle$ is:

$$\Psi = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

Then the coefficient vector in the new basis will be:

$$\tilde{\Psi} = S\Psi = \begin{pmatrix} 3a^* + 7b^* \\ 3a^* - 7b^* \end{pmatrix}$$