E1025: Projector Operators as a Representation Basis

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The problem:

The projection operators P^n and P^{nm} and Q^{nm} on the N^2 states $|n\rangle$, and $|n\rangle + |m\rangle$, and $|n\rangle + i|m\rangle$ respectively, form a complete basis for the representations of operators in the N-dimensional Hilbert space. We also define the generalized Pauli matrices:

$$X^{nm} = 2P^{nm} - P^n - P^m$$

$$Y^{nm} = 2Q^{nm} - P^n - P^m$$

Consider the matrix $\mathcal{A}$ which is written in the standard basis $|n\rangle$ as:

$$\mathcal{A} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 7 & 2 & i \\ 0 & 2 & 5 & 0 \\ 0 & -i & 0 & 9 \end{pmatrix}$$

(1) Explain how the P^n projectors, together with the generalized Pauli matrices can be used as a *convenient* basis. Express $\mathcal{A}$ in this basis.
(2) Express $\mathcal{A}$ as a linear combination of the projection operators as defined above.

The solution:

1. The projection operators are represented in the standard basis $\{|n\rangle\}$ by:

$$P^n = |n\rangle\langle n|$$

$$P^{nm} = \frac{1}{\sqrt{2}}(|n\rangle + |m\rangle)\frac{1}{\sqrt{2}}(\langle n| + \langle m|) = \frac{1}{2}(|n\rangle\langle n| + |m\rangle\langle m| + |n\rangle\langle m| + |m\rangle\langle n|)$$

$$Q^{nm} = \frac{1}{\sqrt{2}}(|n\rangle + i|m\rangle)\frac{1}{\sqrt{2}}(\langle n| - i\langle m|) = \frac{1}{2}(|n\rangle\langle n| + |m\rangle\langle m| - i|n\rangle\langle m| + i|m\rangle\langle n|)$$

For the 4-dimentional space, there are 4 different P^n operators, 6 different P^{nm}, and 6 different Q^{nm}, making a total of $N^2 = 16$ basis operators.

While the P^n operators have a single diagonal element, the P^{nm} and the Q^{nm} operators have both diagonal and off-diagonal elements. Thus, it is convenient to define a new basis for the N^2 representation:

$$P^n = |n\rangle\langle n|$$

$$X^{nm} = 2P^{n,m} - P^n - P^m = |n\rangle\langle m| + |m\rangle\langle n|$$

$$Y^{nm} = 2Q^{n,m} - P^n - P^m = -i|n\rangle\langle m| + i|m\rangle\langle n|$$

In this basis the diagonal and the off-diagonal elements are separated; hence, the matrix $\mathcal{A}$ can be easily decomposed as:

$$\mathcal{A} = 3P^1 + 7P^2 + 5P^3 + 9P^4 + 2X^{2,3} - Y^{2,4}$$

2. The representation in the $\{P^n, X^{nm}, Y^{nm}\}$ basis can be converted to the $\{P^n, P^{n,m}, Q^{n,m}\}$ basis by substituting back the definitions of X^{nm} and Y^{nm}:

$$\begin{aligned}\mathcal{A} &= 3P^1 + 7P^2 + 5P^3 + 9P^4 + 2(2P^{2,3} - P^2 - P^3) - (2Q^{2,4} - P^2 - P^4)\\ &= 3P^1 + 6P^2 + 3P^3 + 10P^4 + 4P^{2,3} - 2Q^{2,4}\end{aligned}$$