E1025: Projector Operators as a Representation Basis

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`Improved phrasing of the problem by DC + remarks`

The problem:

The projection operators P^n and P^{nm} and Q^{nm} on the N^2 states $|n\rangle$, and $|n\rangle + |m\rangle$, and $|n\rangle + i|m\rangle$ respectively, form a complete basis for the representations of operators in the N-dimensional Hilbert space. We also define the generalized Pauli matrices:

$$X^{nm} = 2P^{nm} - P^n - P^m$$

$$Y^{nm} = 2Q^{nm} - P^n - P^m$$

Consider the matrix $\mathcal{A}$ which is written in the standard basis $|n\rangle$ as:

$$\mathcal{A} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 7 & 2 & i \\ 0 & 2 & 5 & 0 \\ 0 & -i & 0 & 9 \end{pmatrix}$$

(1) Explain how the P^n projectors, together with the generalized Pauli matrices can be used as a *convenient* basis. Express $\mathcal{A}$ in this basis.
(2) Express $\mathcal{A}$ as a linear combination of the projection operators as defined above.

The solution:

`DC: missing` $1/\sqrt{2}$ `below.`

1. The projection operators are represented in the standard basis $\{|n\rangle\}$ by:

$$P^n = |n\rangle\langle n|$$

$$P^{nm} = (|n\rangle + |m\rangle)(\langle n| + \langle m|) = |n\rangle\langle n| + |m\rangle\langle m| + |n\rangle\langle m| + |m\rangle\langle n|$$

$$Q^{nm} = (|n\rangle + i|m\rangle)(\langle n| - i\langle m|) = |n\rangle\langle n| + |m\rangle\langle m| - i|n\rangle\langle m| + i|m\rangle\langle n|$$

For the 4-dimentional space, there are 4 different P^n operators, 6 different P^{nm}, and 6 different Q^{nm}, making a total of $N^2 = 16$ basis operators.

`DC: At this point it is useful to define the matrices` X^{nm} `and` Y^{nm} `and` P^n`. Then it is very easy to express` A `using thses matrices. By inspection the result should be:`

$$A = 3P^1 + 7P^2 + 5P^3 + 9P^4 + 2X^{2,3} - Y^{2,4}$$

`DC: The` X `and` Y `matrices are easily expressed uisning the projectors. Results in the next page should be corrected.`

Due to the orthonormality relations of the $\{|n\rangle\}$ basis, the matrix elements of the projection operators are either 1 or $\pm i$, with the diagonal elements always taking the value 1:

$$\langle k|P^{n,im}|k'\rangle = \langle k|n\rangle\langle n|k'\rangle + \langle k|m\rangle\langle m|k'\rangle - i\langle k|n\rangle\langle m|k'\rangle + i\langle k|m\rangle\langle n|k'\rangle$$

Since $\mathcal{A}$ is Hermitian, it's easy to find the representation in the basis of the projection operators by inspection: the symmetrical real and imaginary off-diagonal elements are matched to the multiples of the operators $P^{n,m}$ and $P^{n,im}$, respectively, while the diagonal elements are added a -1 for each overlap with the former operators, and are then matched to multiples of the operators P^n.

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$$\mathcal{A} = P^{4,i2} + 2P^{2,3} + 3P^1 + 4P^2 + 3P^3 + 8P^4$$

2. The representation in the projection operators is easily converted to representation by the generalized Pauli matrices by using the equation which defines the matrices:

$$\sigma_n = 2P^n - P^n - P^n = P^n$$

$$\sigma_{n,m} = 2P^{n,m} - P^n - P^m$$

$$\sigma_{n,im} = 2P^{n,im} - P^n - P^m$$

Thus, the representation in the generalized Pauli matrices is calculated from the result of part 1:

$$\begin{aligned}
\mathcal{A} &= P^{4,i2} + 2P^{2,3} + 3P^1 + 4P^2 + 3P^3 + 8P^4 \\
&= (P^{4,i2} - \frac{1}{2}P^4 - \frac{1}{2}P^2) + (2P^{2,3} - P^2 - P^3) + 3P^1 + 4P^2 + 3P^3 + 8P^4 \\
&= \frac{1}{2}\sigma_{4,i2} + \sigma_{2,3} + 3\sigma_1 + \frac{11}{2}\sigma_2 + 4\sigma_3 + \frac{17}{2}\sigma_4
\end{aligned}$$