Ex1020: Pauli Matrices

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The problem:
Pauli Matrices, including the Identity Matrix, are defined as:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli Matrices and the Identity Matrix together are the complete basis of all 2×2 matrices.

We can define the representation of the given operator A by $A = \sum_i a_i \sigma_i$.

(1) Write these matrices using Dirac notation.

(2) Express $Tr\,(\sigma_k A)$ and $Tr\,(A)$ with a_i.

(3) Generalize the formula for general case: calculate $Tr\,(AB)$ of two operators.

The solution:

(1) The 2×2 matrices are two dimensional matrices. Therefore, we define two orthogonal states (vectors):

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let us use the Dirac notation for writing matrices:

$$A = \sum_{i,j} |i\rangle A_{ij} \langle j|$$

So we can use this for writing Pauli matrices:

$$\sigma_0 = |1\rangle \langle 1| + |2\rangle \langle 2|$$

$$\sigma_1 = |1\rangle \langle 2| + |2\rangle \langle 1|$$

$$\sigma_2 = -i\,|1\rangle \langle 2| + i\,|2\rangle \langle 1|$$

$$\sigma_3 = |1\rangle \langle 1| - |2\rangle \langle 2|$$

(2) At the beginning we express $Tr\left(A\right)$:

$$Tr\left(A\right) = \left\langle 1\right| A \left|1\right\rangle + \left\langle 2\right| A \left|2\right\rangle = \sum_i a_i \left(\left\langle 1\right| \sigma_i \left|1\right\rangle + \left\langle 2\right| \sigma_i \left|2\right\rangle\right) = a_0 + a_0 + a_3 - a_3 = 2a_0$$

Let us represent the multiplication of Pauli matrices by Pauli matrices:

$$\sigma_i \sigma_j = I\delta_{ij} + i\varepsilon_{ijk}\sigma_k$$

Now we can use those multiplications in the next expression:

$$Tr\left(\sigma_k A\right) = \sum_i a_i \left(\left\langle 1\right| \sigma_k \sigma_i \left|1\right\rangle + \left\langle 2\right| \sigma_k \sigma_i \left|2\right\rangle\right)$$

So we get:

$$Tr\left(\sigma_k A\right) = Tr\left(\sigma_k \sum_i a_i \sigma_i\right) = Tr\left(\sum_i a_i \sigma_k \sigma_i\right) = \sum_i a_i Tr\left(\sigma_k \sigma_i\right) = a_k Tr\left(\sigma_k \sigma_k\right) = 2a_k$$

(3) With the replacing σ_k by the general operator B easily we can see that:

$$Tr\left(AB\right) = \sum_{i,j} a_i b_j Tr\left(\sigma_i \sigma_j\right) = 2 \cdot \sum_i a_i b_i$$