

## Ex102: Pauli Matrixes

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### The problem:

The Pauli Matrixes are defined as:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where  $\sigma_0$  is the unit matrix.

(1) Write these matrixes using Dirac notation, choose a base in which  $\sigma_3$  is diagonaled.

We defined:  $X = \sum_{i=0}^3 a_i \sigma_i$

(2) Express  $\text{Tr}(\sigma_i X)$  and  $\text{Tr}(X)$  with  $a_i$ .

(3) Show (with the help of the Dirac formalizm) that the pauli matrixes along with the unit matrix can be used as a base for the  $2 \times 2$  Hermitian matrixes.

### The solution:

(1) The  $2 \times 2$  matrixes are two diamentional matrixes. Therefor, we define two ortogonal states ( vectors):

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To write these matrixes in Dirac notation we will use:

$$A = \sum_{i,j} |i\rangle A_{ij} \langle j|$$

This formula is good for any matrix A and also for all  $\sigma_k$ , therefor:

$$\sigma_0 = |1\rangle 0 \langle 2| + |2\rangle 0 \langle 1| + |1\rangle 1 \langle 1| + |2\rangle 1 \langle 2| = |1\rangle \langle 1| + |2\rangle \langle 2|$$

$$\sigma_1 = |1\rangle 1 \langle 2| + |2\rangle 1 \langle 1| + |1\rangle 0 \langle 1| + |2\rangle 0 \langle 2| = |1\rangle \langle 2| + |2\rangle \langle 1|$$

$$\sigma_2 = |1\rangle - i \langle 2| + |2\rangle i \langle 1| + |1\rangle 0 \langle 1| + |2\rangle 0 \langle 2| = -i |1\rangle \langle 2| + i |2\rangle \langle 1|$$

$$\sigma_3 = |1\rangle 1 \langle 1| + |2\rangle - 1 \langle 2| + |1\rangle 0 \langle 2| + |2\rangle 0 \langle 1| = |1\rangle \langle 1| - |2\rangle \langle 2|$$

Also, we can see that  $\sigma_3$  is already diagonaled so the base for it is the same base we chose before. which is:

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(2) First, the multiplication table of the  $\sigma_i$  ( $i=1,2,3$ ) is:

$$\sigma_i \sigma_j = I \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

which means (for example):

$$\sigma_1 \sigma_2 = \sigma_3 ; \sigma_i^2 = I ; \sigma_0 \sigma_i = \sigma_i$$

Its also easy to see that

$$Tr(\sigma_0) = 2 ; Tr(\sigma_i) = 0 , i = 1, 2, 3$$

Now, with the definition of X we get:

$$Tr(\sigma_0 X) = Tr(\sigma_0 \sum_{i=0}^3 a_i \sigma_i) = Tr[\sigma_0(a_0 \sigma_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3)] = Tr(a_0 \sigma_0 \sigma_0) = a_0 Tr(\sigma_0) = 2a_0$$

We will find all others in the same way. So shortly:

$$Tr(\sigma_i X) = a_i Tr(\sigma_i \sigma_i) = a_i Tr(\sigma_0) = 2a_i , i = 0, 1, 2, 3$$

(3) We know that all Hermitian  $2 \times 2$  matrixes can be written as a linear combination of four ortogonal matrixes. Also, the Pauli matrixes  $\sigma_i$  and the unit  $2 \times 2$  matrix are all ortogonal. Therefor, all four matrixes can be used as a base for the  $2 \times 2$  hermitian matrix.

Just for checking, we will take a general hermitian  $2 \times 2$  matrix and check its Coefficients and whether we can write that matrix as a linear combination of  $\sigma_i$  and the unit matrix:

The general hermitian  $2 \times 2$  matrix is:

$$\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$$

where a,b,c and d real.

Also we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \frac{a}{2}(I + \sigma_3) , \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \frac{d}{2}(I - \sigma_3) , \begin{pmatrix} 0 & b+ic \\ b-ic & 0 \end{pmatrix} = \frac{1}{2}(b\sigma_1 + c\sigma_2)$$

Therefor, we can write the general  $2 \times 2$  Hermitian matrix as follows:

$$\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix} = \frac{1}{2}(a+d)I + \frac{1}{2}b\sigma_1 + \frac{1}{2}c\sigma_2 + \frac{1}{2}(a-d)\sigma_3$$