Decoherence of a particle in a ring

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Abstract. - We consider a particle coupled to a dissipative environment and derive a perturbative formula for the dephasing rate based on the purity of the reduced probability matrix. We apply this formula to the problem of a particle on a ring, that interacts with a dirty metal environment. At low but finite temperatures we find a dephasing rate $\propto T^{3/2}$, and identify dephasing lengths for large and for small rings. These findings shed light on recent Monte Carlo data regarding the effective mass of the particle. At zero temperature we find that spatial fluctuations suppress the possibility of having a power law decay of coherence.

Introduction. – The problem of dephasing of a particle coupled to a dissipative environment at temperature T, and in particular in the limit $T \to 0$ has fascinated the mesoscopic community during the last two decades [1–7]. It has been shown [10,11] that the Caldeira-Leggett (CL) framework [8,9] can be generalized and that the proper way to characterize the environment is by its form factor $\tilde{S}(q,\omega)$. Application of the Feynman-Vernon formalism [10,11] and a semiclassical analysis have shown that an interference amplitude P_φ decays with time as $P_\varphi = \exp(-p_\varphi(t))$ with

$$p_\varphi(t) \;=\; t\int_q\int_\omega \tilde{S}(\boldsymbol{q},\omega)\,\tilde{P}(-\boldsymbol{q},-\omega) \qquad (1)$$

where the integration measures over the wavevector and the frequency are $d^3q/(2\pi)^3$ and $d\omega/2\pi$ respectively. In the semiclassical treatment $\tilde{S}(q,\omega)$ is the *symmetrized* form factor of the environment and $\tilde{P}(q,\omega)$ is the classical *symmetric* power spectrum of the motion. It has been conjectured using add-hock argumentation [12] (see also [2, 13]) that "the correct" procedure is to use non-symmetrized spectra. We expand on this issue further below.

During the last decade the study of a particle in a ring coupled to a variety of environments, has become a paradigm for the study of ground state anomalies [14–18]. Besides being a prototype model problem it may be realized as a mesoscopic electronic device, and it is also of relevance to experiments with cold atoms or ions that are trapped above an "atom chip" device [19–21], where noise is induced by nearby metal surfaces. A significant progress has been achieved in analyzing the equilibrium properties of this prototype system, in particular the dependence of the ground state energy on the Aharonov Bohm flux through the ring.

In the present work we define "dephasing" as the *progressive loss of purity* and find a consistent revised form of Eq.(1) that is valid beyond the semiclassical context. We apply this result to the model of a particle on a ring that interacts with a dirty metal environment. At finite temperature we identify the dephasing rate $\Gamma_\varphi = p_\varphi(t)/t$, that vanishes at zero temperature. At T=0 we find that *only* in the CL-like limit of our model there is still slow progressive spreading ($p_\varphi(t) \sim \ln t$) which suggests a power law decay of coherence. Our results shed new light on recent Monte Carlo data for the temperature dependence of mass-renormalization [22].

Purity. – Our starting point is the most natural definition for the dephasing factor as related to the purity trace(ρ^2) of the reduced probability matrix. The notion of purity is very old, but in recent years it has become very popular due to the interest in quantum computation [23]. Assume that the state of the system including the environment is Ψ_{pn}, where p and n label the basis states of the particle and the bath respectively. Tracing the environment states n defines a reduced probability matrix $[\rho_{sys}]_{p,p'} = \sum_n \Psi_{pn}\Psi^*_{p'n}$ and the purity is then measured by $P_\varphi = \sqrt{\text{trace}(\rho^2_{sys})}$. Consider a factorized initial preparation $\Psi^{(0)}_{pn} = \delta_{p,p_0}\delta_{n,n_0}$, so that within perturbation theory all Ψ_{pn} are small except for $\Psi_{p_0n_0}$. We can relate P_φ to the probabilities $P_t(p,n|p_0,n_0) = |\Psi_{pn}|^2$ to have a transition from the state $|p_0,n_0\rangle$ to the state $|p,n\rangle$ after time t. To leading order we find

$$\begin{aligned} P_\varphi &= P_t(p_0,n_0|p_0,n_0) \\ &+ \sum_{p\neq p_0} P_t(p,n_0|p_0,n_0) + \sum_{n\neq n_0} P_t(p_0,n|p_0,n_0) \end{aligned} \quad (2)$$

The first term in Eq.(2) is just the survival probability P_{survival} of the preparation. The importance of the two other terms can be demonstrated using simple examples: For an environment that consists of static scatterers we have $P_{\text{survival}} < 1$ but $P_\varphi = 1$ thanks to the second term. For a particle in a ring that interacts with a $q = 0$ environmental mode $P_{\text{survival}} < 1$ but $P_\varphi = 1$ thanks to the third term. Using $\sum_{p,n} P_t(p,n|p_0,n_0) = 1$ we finally obtain

$$p_\varphi = 1 - P_\varphi = \sum_{p\neq p_0}\sum_{n\neq n_0} P_t(p,n|p_0,n_0) \quad (3)$$

The dephasing factor p_φ has then the form of a Fermi-golden-rule (FGR), i.e. it is the probability that both the system and the bath make a transition. This differs from the usual FGR treatment [2] in which terms like $P_t(p_0, n \neq n_0|p_0,n_0)$ are included. In the problem that we consider in this paper we can calculate P_φ using a $dqd\omega$ integral as in Eq.(1). In many examples the $\omega = 0$ transitions have zero measure and therefore P_φ is practically the same as $P_t(p_0,n_0|p_0,n_0)$. Otherwise one has to be careful in eliminating those transitions that do not contribute to the dephasing process. Anticipating the application of Eq.(1) for the calculation of the dephasing for a particle in a ring, the integration over q becomes a discrete summation where the q=0 related component should be excluded. It is implicit in the derivation of Eq.(1) from Eq.(3) that at the last step a thermal average is taken over both n_0 and p_0, though in general one may consider non-equilibrium preparations as well.

Dephasing formula. – We would like to apply our revised FGR Eq.(3) to the general problem of a particle at position $\mathbf{R}$ coupled to an environment with electronic density $\mathsf{n}(\mathbf{r},t)$. It is implicit that the particle also experiences an external potential that defines the confining geometry. For definiteness we use the Coulomb interaction, though any other interaction may be used, hence

$$V_{\text{int}} = \int d^3r \rho(\mathbf{r}) \int d^3r' \frac{e^2 \mathsf{n}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \equiv \int d^3r' \rho(\mathbf{r}) \mathcal{U}(\mathbf{r}) \quad (4)$$

where $\rho(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{R}(t))$, with $\mathbf{R}(t)$ the position operator of the particle in the Heisenberg (interaction) picture. Our FGR with $P_t(p,n|p_0,n_0) = |\langle p,n| \int_0^t V_{\text{int}} dt' |p_0,n_0\rangle|^2$ yields

$$\begin{aligned} p_\varphi &= \frac{e^2}{\hbar^2} \sum_{p(\neq p_0)} \sum_{n(\neq n_0)} \int_0^t dt' \int_0^t dt'' \int_r \int_{r'} \\ &\langle p_0|\rho(\mathbf{r}'',t'')|p\rangle\langle p|\rho(\mathbf{r}',t')|p_0\rangle \\ &\langle n_0|\mathcal{U}(\mathbf{r}'',t'')|n\rangle\langle n|\mathcal{U}(\mathbf{r}',t')|n_0\rangle \end{aligned} \quad (5)$$

The double time integral can be written as a $dqd\omega$ integral over Fourier components. For this purpose we define the form factor of the *fluctuations* (as seen by the particle):

$$\tilde{S}(\boldsymbol{q},\omega) = \int d^3r \int d\tau \, \langle \mathcal{U}(\mathbf{r}',t')\mathcal{U}(\mathbf{r},t)\rangle \, \mathrm{e}^{i\omega\tau - i\boldsymbol{q}\cdot\mathbf{r}} \quad (6)$$

with thermal average replacing the n_0 state expectation value. $\tilde{S}(q,\omega)$ is related to the dielectric function of the environment $\varepsilon(\boldsymbol{q},\omega)$ via the fluctuation dissipation theorem

$$\tilde{S}(\boldsymbol{q},\omega) = \frac{4\pi e^2}{\boldsymbol{q}^2} \mathrm{Im}\left[\frac{-1}{\varepsilon(\boldsymbol{q},\omega)}\right] \frac{2}{1-\mathrm{e}^{-\omega/T}} \quad (7)$$

In the semiclassical formulation one replaces the operator $\boldsymbol{R}(t)$ by the classical trajectory $\boldsymbol{R}_{cl}(t)$, and consequently in Eq.(5) the particle-related part of the integrand is replaced by a classical two-point correlation function of the type $\langle f(\boldsymbol{R}_{cl}(t''))f(\boldsymbol{R}_{cl}(t'))\rangle$. In the quantum context the analogous treatment after Fourier transform leads to the following definition for the power spectrum of the motion:

$$\tilde{P}(\boldsymbol{q},\omega) = \int \left[\langle \mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{R}(\tau)} \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{R}(0)}\rangle - \langle \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{R}}\rangle^2\right] \mathrm{e}^{i\omega\tau}\, d\tau \quad (8)$$

Also here, close to equilibrium condition, a thermal average should replace the p_0 state expectation value. It is important to realize that this definition, as well as Eq.(7), imply that non-symmetrized spectral functions should be used. Our main interest is in very low temperatures, so we set for presentation purpose $p_0 = 0$. Then we get

$$\tilde{P}(\boldsymbol{q},\omega) = \sum_p |\langle p|\mathrm{e}^{i\boldsymbol{q}\cdot\mathbf{R}}|0\rangle|^2 \, \delta(\omega - E_p) \quad (9)$$

Using the above *definitions* for $\tilde{S}(\boldsymbol{q},\omega)$ and $\tilde{P}(\boldsymbol{q},\omega)$ we can re-write

$$\begin{aligned} p_\varphi = \int_0^t dt' \int_0^t dt'' \int_q \iint_{\omega',\omega''} &\tilde{S}(\boldsymbol{q},\omega')\tilde{P}(-\boldsymbol{q},-\omega'') \\ &\times \mathrm{e}^{-i(\omega'-\omega'')(t''-t')} . \end{aligned} \quad (10)$$

For purely aesthetic reasons we prefer to use soft rather than sharp cutoff for the time integration. Then we get

$$p_\varphi = t \int_q \iint_{\omega,\omega'} \tilde{S}(\boldsymbol{q},\omega)\tilde{P}(-\boldsymbol{q},-\omega') \left[\frac{(4/t)}{(2/t)^2 + (\omega-\omega')^2}\right] \quad (11)$$

This result can be cast into the form of Eq.(1) provided $\tilde{P}(\boldsymbol{q},\omega)$ is *re-defined* as the convolution of Eq.(8) with the kernel in the square brackets, which is like time-uncertainty broadened delta function.

Dephasing rate. – At finite temperatures, if t is larger compared with dynamically relevant time scales, and in particular $t \gg \hbar/T$, we can replace the square brackets in Eq.(11) by $2\pi\delta(\omega-\omega')$. Consequently we get linear growth $p_\varphi \approx \Gamma_\varphi t$ with the rate

$$\Gamma_\varphi = \int_q \int_\omega \tilde{S}(\boldsymbol{q},\omega)\tilde{P}(-\boldsymbol{q},-\omega) \quad (12)$$

Following standard arguments one conjectures that the long time decay is exponential, i.e. $P(t) = \exp(-\Gamma_\varphi t)$, as in the analysis of Wigner's decay. In terms of the dielectric function $\varepsilon(q,\omega)$ we obtain the following general result:

$$\Gamma_\varphi = \sum_{p\neq 0} \int_q \frac{4\pi e^2}{q^2} \mathrm{Im}\left[\frac{1}{\varepsilon(q,-E_p)}\right] \frac{2|\langle p|\mathrm{e}^{i\mathbf{q}\cdot\mathbf{R}}|0\rangle|^2}{\mathrm{e}^{E_p/T}-1} \quad (13)$$

Our assumption $t \gg (1/T)$ implies that (13) can be trusted only if $\Gamma_\varphi \ll T$ which implies a weak coupling condition (see below).

Dirty metal. – So far we kept the derivation general, without specifying either the particle states $|p\rangle$ or the dielectric function $\varepsilon(q,\omega)$. We now consider the effect of low frequency fluctuations ($|q| \lesssim 1/\ell, |\omega| \lesssim \omega_c$) due to a dirty metal environment for which $\varepsilon(q,\omega) = 1 + 4\pi\sigma(-i\omega + Dq^2)^{-1}$, where σ is the conductivity, D is the diffusion constant, and ℓ is the mean free path. Below we identify the renormalized value of the high frequency cutoff ω_c as the classical damping rate $\gamma_r = 2\pi\alpha/M\ell^2$, where the dimensionless interaction strength is $\alpha = e^2/(8\pi^2\sigma\ell) = 3/(8(k_F\ell)^2)$ and k_F is the Fermi wavevector. Furthermore, we consider a particle on a ring of radius R, where $|p_m\rangle \propto \mathrm{e}^{im\theta}$ are the eigenstates with energies $E_m = m^2/(2MR^2)$. We first consider the case of a large ring with $r = R/\ell \gg 1$. Using the Fourier expansion [16, 17]

$$\ell \int_q \mathrm{e}^{-i\mathbf{q}\cdot(\mathbf{R}(\theta)-\mathbf{R}(\theta'))} \frac{4\pi}{q^2} = \frac{1}{\sqrt{4r^2\sin^2(\frac{\theta-\theta'}{2})+1}}$$
$$= 1 - \sum_m a_m \sin^2\left(\frac{m(\theta-\theta')}{2}\right) \quad (14)$$

with

$$a_m \approx \begin{cases} \frac{2}{\pi r}\ln\frac{r}{m}, & 1 \leq m \leq r \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

we have

$$\int_q \frac{4\pi}{q^2} |\langle 0|e^{-i\mathbf{q}\cdot\mathbf{R}}|p_m\rangle|^2 = \frac{1}{4} a_{|m|} \quad (16)$$

and therefore

$$\Gamma_\varphi = 2\pi\alpha \sum_{m\neq 0} a_m \frac{E_m \mathrm{e}^{-|E_m|/\omega_c}}{\mathrm{e}^{E_m/T}-1} \approx 2\pi\alpha T \sum_{0<|m|<r_{\text{eff}}} a_m \quad (17)$$

where $r_{\text{eff}} \equiv \min\{r, (2MR^2T)^{1/2}, (2MR^2\omega_c)^{1/2}\}$ is determined by the conditions $m < r$ and $E_m < T$ and $E_m < \omega_c$. At high temperatures r_{eff} is temperature independent and therefore $\Gamma_\varphi \propto T$, while at low temperatures [but still $T > 1/(2MR^2)$] we get, as shown schematically in Fig. 1,

$$\Gamma_\varphi = 4\alpha T\sqrt{2M\ell^2T}|\ln\sqrt{2M\ell^2T}| \sim T^{3/2}|\ln T| \quad (18)$$

From these results it follows that the self consistency requirement $\Gamma_\varphi \ll T$, as discussed after Eq.(13), is globally satisfied for any temperature if $\alpha \ll 1$, or equivalently if $k_F\ell \gg 1$. We note that if the particle were diffusing in a disordered media, then a momentum cutoff $\sim T^{1/2}$ would also result in a $\sim T^{3/2}$ behavior. We also note that if the m=0 Fourier component were included in the summation, then the low T form would change to $\Gamma_\varphi \propto T$, in contrast with the proper result Eq.(18).

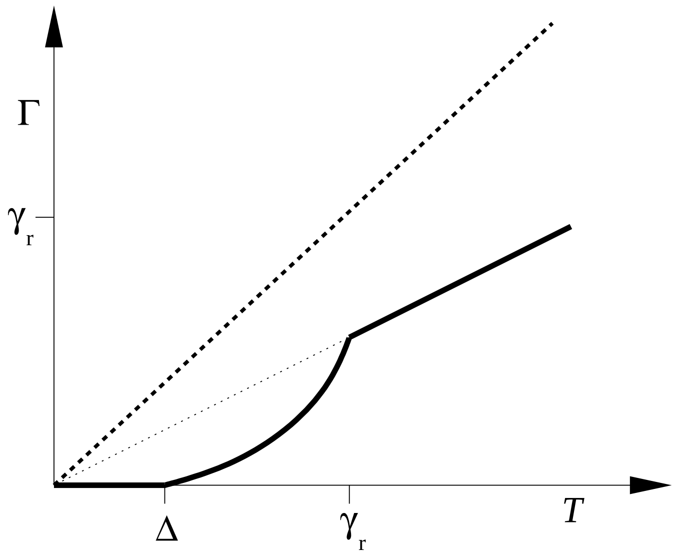

Fig. 1: Illustration of the dependence of the dephasing rate Γ on the temperature T. The dephasing rate is well defined for $t > (1/T)$, and hence the self consistency requirement is $\Gamma \ll T$. This condition is demonstrated by a comparison with the dashed line. The illustration assumes weak coupling $\alpha \ll 1$ and large rings $\alpha r^2 \gg 1$ so that the energy cutoff is $\gamma_r = 2\pi\alpha/M\ell^2 \gg \Delta = 1/MR^2$. For extremely low temperatures, such that T is smaller compared with the spacing Δ, the probability to excite the system is exponentially small and the familiar two-level modeling becomes applicable.

Zero temperature. – We would like to discuss the "zero temperature" regime. If the temperature were extremely low, such that $T \ll \Delta$ where $\Delta \sim 1/(MR^2)$ is the ground state level spacing, we could treat the problem using a "two level approximation", which is a very well studied model [24]. We do not further discuss this regime. From here on we assume $T \gg \Delta$. Thus we can treat the $d\omega$ integration as if the levels of the ring form a continuum. But we still can define "zero temperature" as such for which the practical interest is in the time interval $t \ll 1/T$, which can be extremely long Then one realizes that Eq.(11) gives a non-zero result even at "zero temperature":

$$p_\varphi \approx \alpha \sum_m a_m \ln \frac{\omega_c}{E_m + (1/t)} \quad (19)$$

where ω_c is the high frequency cutoff of the environmental modes. Assuming $\ell \ll R$ we get after a transient:

$$p_\varphi \approx \alpha\ell \int_0^{1/\ell} dq \, \ln\left[\frac{1}{q\ell}\right] \ln\left[\frac{M\omega_c}{q^2}\right] \approx \alpha \quad (20)$$

where for clarity of presentation we converted the m summation into a $dq \equiv (1/R)dm$ integral. Accordingly we deduce that for $r \gg 1$ coherence is maintained if $k_F\ell \gg 1$. This should be contrasted with the CL limit ($\ell \to \infty$) where the integral has a singular $q \sim 0$ contribution from the lowest fluctuating mode ($m=1$), and consequently $p_\varphi \approx 2\alpha r^2 \log(\omega_c t)$. The CL does not apply here because we assume weak coupling ($\alpha \ll 1$). Therefore once $\alpha r^2 < 1$, the quantization of the energy spectrum becomes important and the renormalized cutoff frequency becomes $\omega_c \sim \Delta$ instead of $\omega_c \sim \gamma_r$. Accordingly, in the latter case, the time during which the log spreading prevails diminishes.

Effective mass. – It can be shown [10] that the mass renormalization in the inertial (polaronic) sense is $\Delta M = \eta/\omega_c$. However in recent works [16–18] the mass renormalization concept appears in a new context. The free energy $\mathsf{F}(T,\Phi)$ of a particle in a ring is calculated, where Φ is the Aharonov Bohm flux through the ring. Then the coherence is characterized by the "curvature", which is a measure for the sensitivity to Φ. The curvature can be parameterized as

$$\left.\frac{\partial^2 \mathsf{F}}{\partial \Phi^2}\right|_{\Phi=0} = \frac{e^2}{M^* R^2} f(M^* R^2 T) \qquad (21)$$

where in the absence of environment $M^* = M$ is the bare mass of the particle, and the T dependence simply reflects the Boltzmann distribution of the energy. In the presence of coupling to the environment $M^* > M$ and M^* depends on both α and T. At $T=0$, for fixed $\alpha \ll 1$, Monte Carlo data show [22] that the ratio M^*/M is independent of the radius provided $\alpha r^2 \gtrsim 1$. As the radius becomes smaller compared with the implied critical value r_c, the ratio M^*/M rapidly approaches unity. In the regime of "large R" the mass renormalization effect diminishes with the temperature and depends on the scaled variable RT, while for "small R" the ratio M^*/M grows with the temperature, and depends on the scaled variable R^4T. The natural question is whether we can shed some light on the physics behind this observed temperature dependence. Making the conjecture that the temperature dependence of M^*/M is determined by dephasing it is natural to suggest the following measure of coherence:

$$x(T,R) = p_\varphi\left(t{=}\frac{1}{\Delta_{\text{eff}}}\right) \approx \frac{\Gamma_\varphi}{\Delta_{\text{eff}}} \approx 2\pi\alpha\bar{a}MR^2T \qquad (22)$$

where $\bar{a}$ is an average value of a_n. Eq.(22) is the dephasing factor at the time $t = 1/\Delta_{\text{eff}}$, where $\Delta_{\text{eff}} \sim r_{\text{eff}} \times (MR^2)^{-1}$ is the energy scale that characterizes the "effective" transitions; hence the variable x measures the level sharpness. For a dirty metal with $\ell \ll R$ the typical value of the Fourier components is $\bar{a} \sim 1/r$ as implied by Eq.(15). On the other hand for a dirty metal with $\ell \gg R$ there is only one effective mode with $\bar{a} \sim r^2$. Accordingly we get the RT and the R^4T dependence respectively, in agreement with the Monte Carlo simulations. The condition $x > 1/2$ can serve as a practical definition for having coherence. It can be translated either as a condition on the temperature, or optionally it can be used in order to define a coherence length that depends on the temperature. The conjecture is that M^*/M is a function of x.

Summary. – In this paper we derive a new perturbative expression for the dephasing factor $P_\varphi(t)$ and apply it to a particle in a ring coupled to fluctuations of a dirty metal environment. We find that the dephasing rate vanishes at $T=0$. We also define a coherence criterion that identifies a dephasing length. The latter diverges as T^{-1} for large radius and as $T^{-1/4}$ for small radius, in consistency with Monte Carlo data on mass renormalization. The renormalized mass is an equilibrium property which affects temporal correlation functions. But we see that it reflects nonequilibrium features of the dynamics which are expressed in the dephasing factor calculation. We find this relation between equilibrium and nonequilibrium scales an intriguing phenomena.

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