# The theory of localization

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Overview of the "localization" theme adopting a possibly biased personal perspective. Intended as an entertaining "retreat talk" for undergraduate students.

## [1] What does it mean localization?

Simplest is to understand the 1D case.

It is not related necessarily to "quantum mechanics".

$$\text{Stochastic rate equation} \qquad \frac{d}{dt}\mathbf{p} = \mathbf{W}\mathbf{p}, \qquad \mathbf{p} \equiv \text{vector}\{p_n\} \tag{1}$$

$$\text{Resistor network problem} \qquad \frac{d}{dt}\mathbf{Q} = \mathbf{G}\mathbf{V}, \qquad \mathbf{Q} \equiv \text{vector}\{Q_n\}, \quad Q_n = CV_n \tag{2}$$

$$\text{Schrödinger equation} \qquad \frac{d}{dt}\boldsymbol{\psi} = -i\boldsymbol{\mathcal{H}}\boldsymbol{\psi}, \qquad \boldsymbol{\psi} \equiv \text{vector}\{\psi_n\} \tag{3}$$

$$\text{Newton eq for balls+springs} \qquad \frac{d^2}{dt^2}\mathbf{u} = -\mathbf{K}\mathbf{u}, \qquad \mathbf{u} \equiv \text{vector}\{u_n\} \tag{4}$$

Real symmetric banded matrices:

$$\mathbf{W} \equiv \text{diag}\{-\gamma_n\} + \text{offdiag}\{w_{nm}\} = \begin{bmatrix} -\gamma & w & 0 & 0 & ... \\ w & -\gamma & w & 0 & ... \\ 0 & 0 & -\gamma & w & ... \\ 0 & 0 & w & -\gamma & ... \\ ... & ... & ... & ... & ... \end{bmatrix} \tag{5}$$

Conservative if:

$$\gamma_n = \sum_{m \neq n} w_{mn} \tag{6}$$

- Without disorder plane waves $(k)$ are the eigenstates.
- In particular note the $k=0$ eigenstate.
- Balls-connected-by-springs $\rightsquigarrow$ heat-transport problem
- Debye localization versus Anderson localization
- The "resistor network" aspect: glassy disorder, percolation
- Duality: Glassy off-diagonal disorder $\mapsto$ weak diagonal disorder.

`Weinberg, de Leeuw, Kottos, Cohen`,
Resistor-network anomalies in the heat transport of random harmonic chains [arXiv 2016].

## ====== [2] The Anderson criterion for localization

Anderson, Absence of Diffusion in Certain Random Lattices [Phys Rev 1958]

Calculating the Green function for a particle in a disordered lattice. By definition the connective constant $\mu$ determines the number $n^{\mu}$ of self-avoiding paths that start at the origin, where $n \to \infty$ is the path length. The Green function is schematically

$$G \sim \sum_n \mu^n \left(\frac{K}{E-\epsilon}\right)^n \tag{7}$$

where the $\epsilon \in [-W/2, +W/2]$ are random on-site energies, and $K$ is the hopping amplitude. The condition for convergence (implying localization) is

$$\mu \left\langle \frac{K}{E-\epsilon} \right\rangle_{\epsilon} < 1 \tag{8}$$

In the band center it leads to the Anderson criterion

$$\mu \frac{K}{W} \ln\left(\frac{W}{K}\right) < 1 \tag{9}$$

Or one may say roughly that there is localization if $\mu K < W$, which is the condition for not mixing the orbitals by the hopping.

## ====== [3] Scaling theory: 1D/2D/3D

The definition of $g$.

Scaling theory implies a mobility edge for 3D.

## ====== [4] Quantum Localization - breaktime concept

The semiclassical perspective on "weak localization" is known as "scar theory". The breaktime concept has first appeared in the context of "strong" 1D dynamical localization in the kicked-rotor problem, and later generalized in various ways.

Chirikov, Izrailev, Shepelyansky [SovSciRevC 1981], Shepelyansky [PhysicaD 1987]
Heller, Quantum localization and the rate of exploration of phase space [PRA 1987]
Dittrich, Spectral statistics for 1D disordered systems [Phys Rep 1996]
Cohen, Periodic Orbits Breaktime and Localization [JPA 1998]
Cohen, Yukalov, Ziegler, Hilbert-space localization in closed quantum systems [PRA 2016]

The breaktime is determined from the breakdown of the QCC requirement:

$$t \ll t_H[\Omega(t)] \quad \rightsquigarrow \quad t^* \tag{10}$$

The well known semi-classical result for localization in 1D is easily recovered. Setting $\Omega(t) = \sqrt{Dt}$ and $t_H = 2L/v$ we get $t^* = 4D/v^2$, leading to $\xi = 2D/v = 2\ell$. This strict 1D version s a somewhat misleading because an explicit $\hbar$ dependence is absent. The puzzle is solved by noting that in strict 1D the diffusion cannot be of classical origin. If we go to (say) an $f = 2$ quasi-1D billiard system with classical diffusion, the expression for the Heisenberg time is multiplied by an $\hbar$-dependent factor (so called "number of channels") and we deduce that

$$\xi \propto \frac{D_{\text{cl}}}{\hbar} \tag{11}$$

Hence the formula acquires an explicit $\hbar$ dependence as expected.

In 3D we deduce the existence of a mobility edge.

## ====== [5] Dynamical localization - the kicked rotor

The dynamics of the kicked rotor is described by the so-called *standard map*:

$$p_{n+1} = p_n + K\sin(\theta_n) \quad (12)$$

$$\theta_{n+1} = \theta_n + p_{n+1} \quad (13)$$

For $K \gg 1$ we get diffusion because the following is like random walk

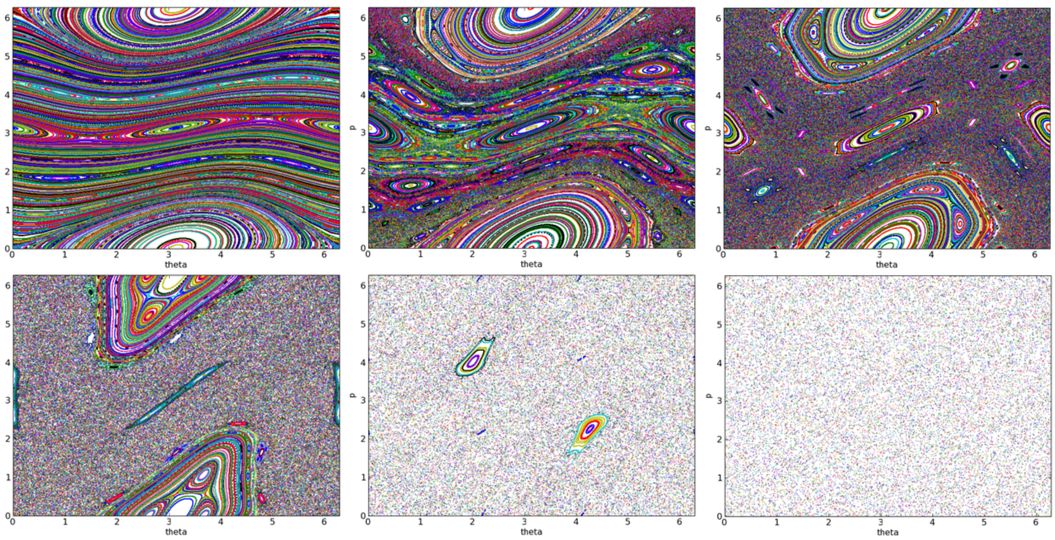
$$p_N - p_0 = K\sin(\theta_1) + K\sin(\theta_2) + \cdots + K\sin(\theta_N) \quad (14)$$

Upon quantization this is a prototype model for dynamical localization.

Here the localization is not related to disorder but to the underlying chaos.

`Casati, Chirikov, Izrailev, J. Ford` [Lecture Notes in Physics 1979]

`Fishman, Grempel, Prange` [PRL 1982]




With noise or "bath" localization is destroyed.

`Ott, Antonsen, Hanson` [PRL 1984]

`Dittrich Graham` [ZPB 1986]

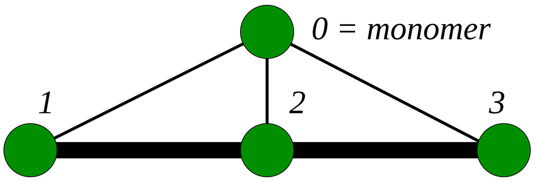
`Cohen` [PRL 1991] [PRA 1991]

## [6] Dynamical localization - many body context

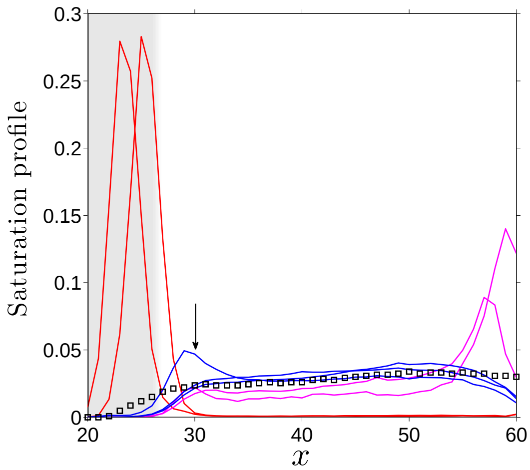
Let us start with a simple example for dynamical localization in a minimal Bose-Hubbard model. The FPE description makes sense if at least one sub-system is chaotic.

- Minimal model for a chaotic sub-system: BHH trimer
- Minimal model for thermalization: BHH trimer + monomer

Khripkov, Vardi, Cohen [NJP 2015]

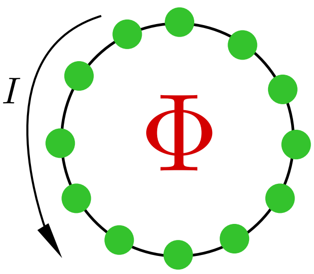


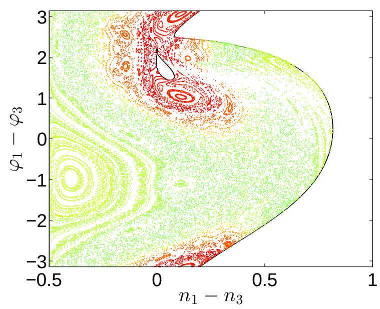

We see that that for some initial conditions we get localization.
For small $x$ preparation the localization is related to phase-space structures, while for large $x$ it is Anderson-type localization in a chaotic sea.

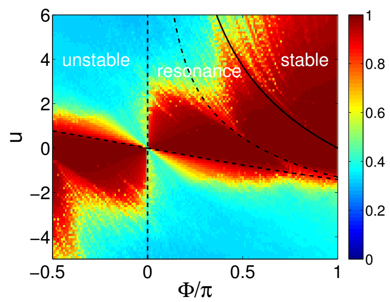



Another example for manifestation of dynamical localization in the BHH context concerns a ring geometry. It is responsible for what we call quantum metastability:

Arwas, Vardi, Cohen, Superfluidity and Chaos in low dimensional circuits [Scientific Reports 2015]



## [7] Disorder and interactions - many body localization?

`Basko, Aleiner, Altshuler` [Annals Phys 2006] [arXiv]
`Gornyi, Mirlin, Polyakov` [PRL 2005]

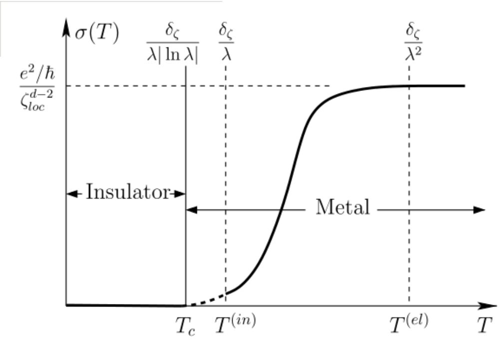
The discussion of MBL has originally appeared in the electronic context (fermions). The role of the disorder $W$ is played by the mismatch of energies that are coupled by the two-body interaction term. Assuming that coupled orbitals reside at the same localization volume

$$\left|\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta\right| \quad \sim \quad \Delta_\xi \tag{15}$$

It is argued that the effective connective constant is $\mu \sim T/\Delta_\xi$. The strength of the interaction (relative to $\Delta_\xi$) is denoted $\lambda$. Hence the Anderson criterion takes the following form

$$\frac{T}{\Delta_\xi}\lambda|\ln\lambda| \quad < \quad 1, \qquad\qquad \lambda = \frac{U}{\Delta_\xi} \tag{16}$$

The implications of MBL on the properties of the sample are important for both numerical and experimental studies. Clearly is does not affect the density of states, hence it is not a thermodynamic phase-transition. But we expect an abrupt change in the conductivity (see schematic plot), and it should be reflected in the sensitivity to twist boundary conditions. It also should affect the level statistic and the $\omega \sim 0$ response characteristics (see later section). Another measure is the segment-size dependence of the entanglement entropy (see later section).




## [8] Hopping due to phonons

The phonons form a bath.

The bath introduces noise.

The noise induces decoherence, aka dephasing.

Hopping conductance.

## ═══ [9] Depahsing due to the "other electrons"

`Altshuler, Aronov, Khmelnitskii`,
Effects of electron-electron collisions with small energy transfers on quantum localisation [JPC 1982]

`Stern, Aharonov, Imry`,
Phase uncertainty and loss of interference [PRA 1990]

$$\text{DephasingRate} \quad \propto \quad T^{3/2} \tag{17}$$

Residual dephasing at zero temperature?

`Golubev ,Zaikin`,
Quantum Decoherence in Disordered Mesoscopic Systems [PRL 1998]

`Cohen, Imry`,
Dephasing at low temperatures [PRB 1999]

`Aleiner, Altshuler, Gershenson`,
Comment on "Quantum Decoherence in Disordered Mesoscopic Systems" [PRL 1999]

`Cohen, Horovitz`,
Decoherence of a particle in a ring [JPA 2007] [EPL 2008]

`Cohen, von Delft, Marquardt, Imry`,
Dephasing rate formula in the many-body context [PRB 2009]

Dephasing at finite temperatures?

The argument against the survival of Anderson localization is as follows: Once inter-particle interaction is introduced, each particle experiences a bath that consists of all the other particles; hence decoherence destroys the Anderson localization; leading to hopping conductivity. The answer to this fallacy is that the local bath has a discrete spectrum; averaging the spectrum over disorder realization is apparently correct for the purpose of calculating dephasing rate, but wrong as far as breaktime is concerned.

Two interacting particles

`Shepelyansky`,
Coherent Propagation of Two Interacting Particles in a Random Potential [PRL 1994]

`Imry`,
Coherent Propagation of Two Interacting Particles in a Random Potential [EPL 1995]

## ═══ [10] Bosons with disorder in 1D

`Aleiner Altshuler Shlyapnikov`,
A finite-temperature phase transition for disordered weakly interacting bosons in one dimension [Nature Phys 2010]

The two challenging statements are:

- existence of a phase transition in one dimension.
- persistence of quantum phase-transition at finite temperature.

Related to the (disorder, inteaction) phase diagram of the disordered Bose-Hubbard model
[Giamarchi, Schulz (PRB 1988)]
[Lugan et al (PRL 2007)]
[Gurarie et al, (PRB 2009)]