Non-equilibrium steady state of “sparse” systems

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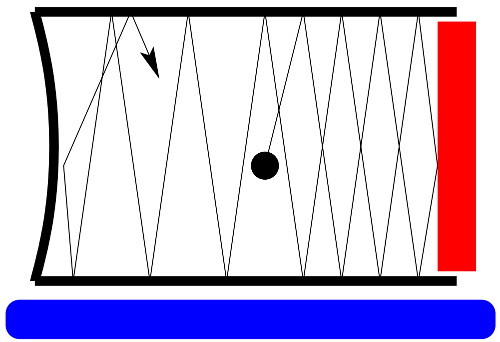
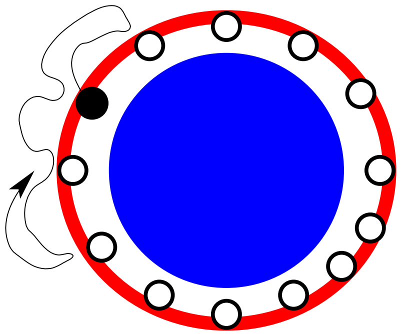
$$\mathcal{H}_{\text{total}} \;=\; \text{diag}\{E_n\} - f(t)V_{nm} + F(t)W_{nm} + Bath$$

Daniel Hurowitz (BGU) [1,2]
Saar Rahav (Technion) [2]

Alex Stotland (BGU) [3,4]
Lou Pecora (NRL) [3]
Nir Davidson (Weizmann) [4]





[1] Hurowitz, Cohen (EPL 2011)
[2] Hurowitz, Rahav, Cohen (EPL 2012)
[3] Stotland, Pecora, Cohen (EPL 2010, PRE 2011)
[4] Stotland, Cohen, Davidson (EPL 2009)

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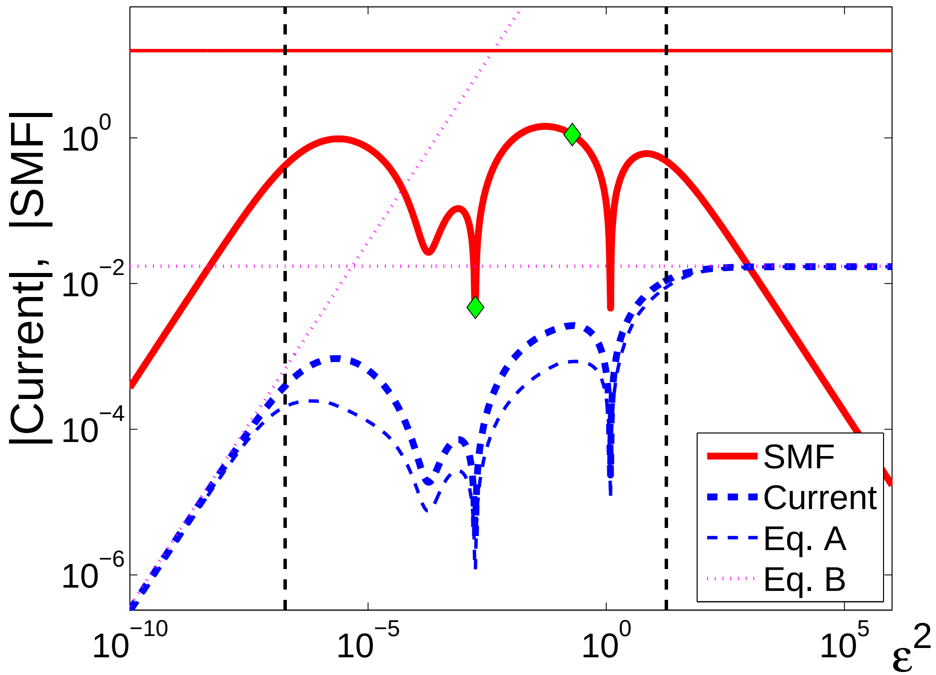
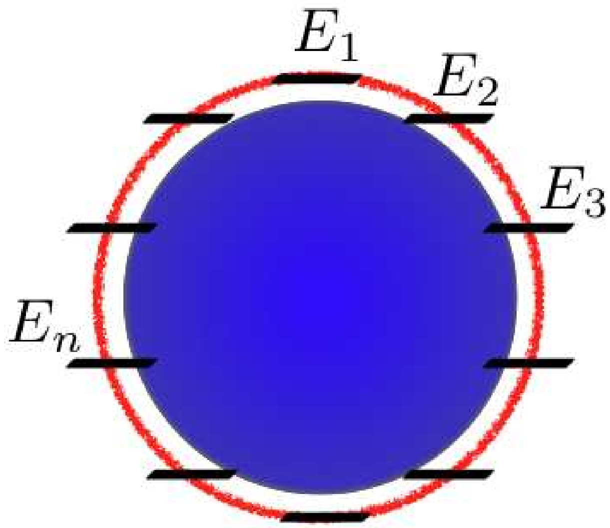
$DIP, $BSF, $ISF

Current versus driving

Driving $\rightsquigarrow$ Stochastic Motive Force $\rightsquigarrow$ Current

Regimes: LRT regime, Sinai regime, Saturation regime





$$I \ \sim \ \frac{1}{N}\, \overline{w} \ \exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right] \ 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

Master equation description of dynamics

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)V_{nm} + F(t)W_{nm} + Bath$$

Quantum master equation for the reduced probability matrix:

$$\frac{d\rho}{dt} = -i[\mathcal{H}_0, \rho] - \frac{\varepsilon^2}{2}[V,[V,\rho]] + \mathcal{W}^{\beta}\rho$$

Corresponding stochastic rate equation:

$$\frac{dp_n}{dt} = \sum_m w_{nm}p_m - w_{mn}p_n$$

$$w_{nm} = w^{\varepsilon}_{nm} + w^{\beta}_{nm}$$

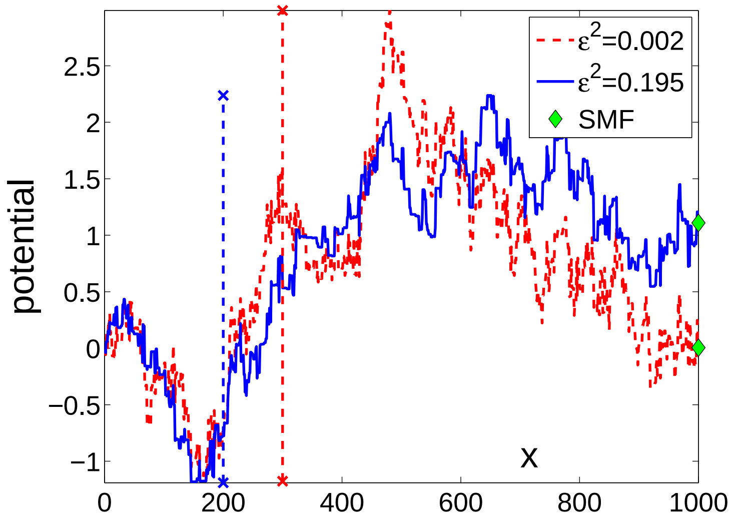
$$w^{\varepsilon}_{nm} = w^{\varepsilon}_{mn} \propto \varepsilon^2$$

$$\frac{w^{\beta}_{nm}}{w^{\beta}_{mn}} = \exp\left[-\frac{E_n - E_m}{T_B}\right]$$

Steady state equation:

$$\dot{\rho} = \mathcal{W}\rho = 0$$

The Stochastic Motive Force



If we had only a bath

$$\frac{w_{nm}}{w_{mn}} = \exp\left[-\frac{E_n - E_m}{T_B}\right]$$

We define a "field"

$$\mathcal{E}(x) \equiv \ln\left[\frac{w_{nm}}{w_{mn}}\right]$$

The "potential" variation along a segment

$$\mathcal{E}(x_1 \rightsquigarrow x_2) = \sum_{x=x_1}^{x_2} \mathcal{E}(x) = \int_{x_1}^{x_2} \mathcal{E}(x)dx$$

$$\mathcal{E}_{\cap} \equiv \text{maximum}\left\{\left|\mathcal{E}(x_1 \rightsquigarrow x_2)\right|\right\}$$

$$\mathcal{E}_{\circlearrowleft} \equiv \oint \mathcal{E}(x)dx \qquad \text{if no driving } = 0$$

The emergence of Sinai regime

Sinai [1982]: Transport in a chain with random transition rates.

Assume transition rates are uncorrelated.

$\rightsquigarrow$ exponential build up of a potential barrier $\mathcal{E}_{\cap} \propto \sqrt{N}$

$\rightsquigarrow$ exponentially small current.

But... we have telescopic correlations: $\mathcal{E}_{n,n+1} \sim \Delta_n \equiv (E_n - E_{n+1})$

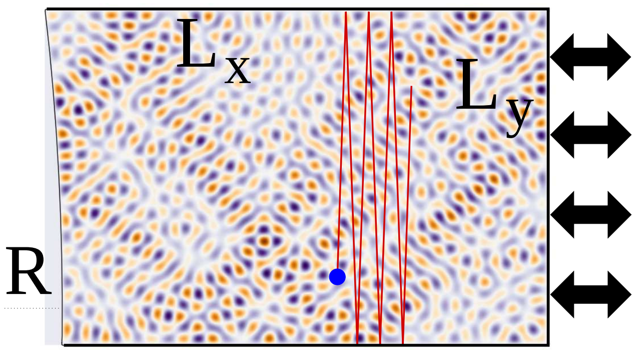
$$\mathcal{E}_{\circlearrowleft} \;\approx\; -\sum_n \left[\frac{1}{1+g_n\epsilon^2}\right]\frac{\Delta_n}{T_B} \;\sim\; \frac{1}{T_B}\begin{cases} \epsilon^2, & \epsilon^2 < 1/g_{\max} \\ 1/\epsilon^2, & \epsilon^2 > 1/g_{\min} \\ [\pm]\sqrt{N}\Delta, & \text{otherwise} \end{cases}$$

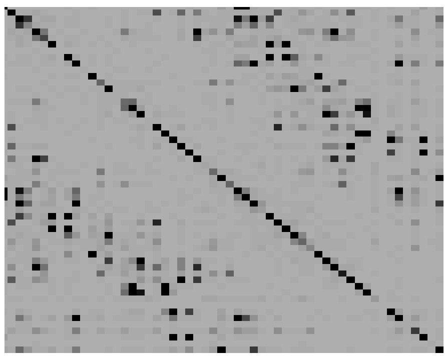
Build up may occur if g_n are from a **log-wide** distribution.

$$I \;\sim\; \frac{1}{N}\,\overline{w}\,\exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right]\,2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

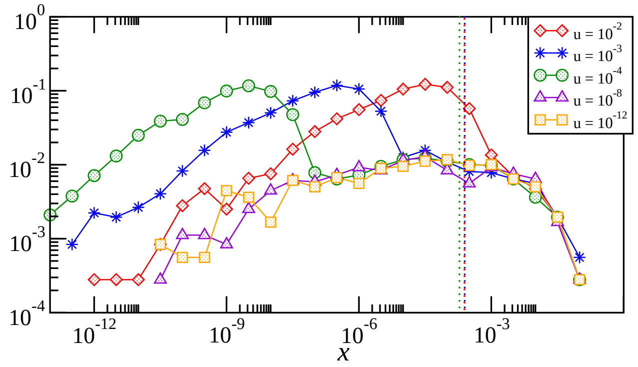
“Sparsity” of weakly chaotic systems

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)V_{nm}$$

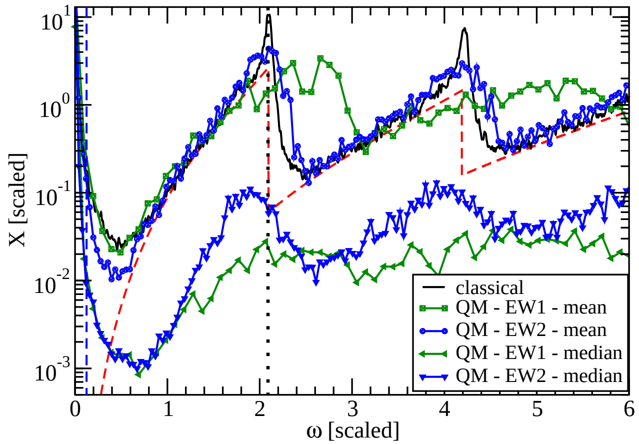




not a Gaussian matrix...



[log-wide distribution]



[median $\ll$ mean]

$$\text{mean}[|V_{nm}|^2] \;\approx\; (2\pi\varrho)^{-1}\;\tilde{C}_{cl}(E_n - E_m)$$

Motivation for this line of study

B The Kubo–Greenwood conductivity and the Edwards–Thouless relationships

Writing (but being aware of the subtleties) the matrix elements of x in terms of those of v, noting, for example, that for positive ω, $\omega = \omega_{ng}$, one expresses the low-frequency real conductivity σ_{xx}, from eq. A.7 inserting the volume of the system, *Vol*, as

$$\mathrm{Re}\,\sigma(\omega) = \frac{\pi e^2}{Vol\,\hbar\omega} \sum_n |v_{gn}|^2 \delta(\omega - \omega_{ng}), \qquad (B.1)$$

where the cartesian index x has been dropped for the diagonal conductivity, $\sigma_{xx} \equiv \sigma$. For noninteracting quasiparticles the excited states are particle–hole excitations where the hole can be created anywhere between ϵ_F and $\epsilon_F - \hbar\omega$. Replacing the $|v_{gn}|^2$ in this small range by an average value $\overline{v^2}$, one obtains, since the excitations are electron-hole ones

$$\sigma \equiv \mathrm{Re}\,\sigma(\omega \to 0) = \frac{\pi e^2 \hbar}{Vol} \overline{v^2} [N(0)]^2, \qquad (B.2)$$

where $N(0)$ is the single-particle density of states per unit energy $N(0) = n(0)$. *Vol*. This is called the Kubo–Greenwood formula. Note that this is valid for a large enough system having effectively a continuous spectrum. How to handle the discrete spectrum in mesoscopics is discussed in chapter 5.

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usual expression for σ from eq. 5.1, one straightforwardly obtains the low-temperature d.c. conductivity by replacing the sums by integrals and assuming that $|\langle l|\hat{v}|k\rangle|^2$ has some typical value denoted by $|\langle v\rangle|^2$ near ϵ_F (see appendix B),

$$\sigma_{KG} = \pi e^2 Vol\,\hbar |\langle v\rangle|^2 [n(0)]^2. \qquad (5.3)$$

Here $n(0)$ is the density of states per unit volume, at ϵ_F. This is the Kubo–Greenwood conductivity (Kubo 1957, Greenwood 1958).

Appendix + Chapter5 from the book of Joe Imry

Question:

How to go beyond linear response?

- Average value?
- Typical value?

Answer:

"Resistor network average"

$$G \;=\; \pi\varrho^2 \,\langle\langle |V_{nm}|^2 \rangle\rangle$$

Estimate:

"Effective range hopping"

Applications:

Heating of cold atoms due to vibrating walls

[beyond the "wall formula" of Swiatecki]

Mesoscopic conductance of EMF-driven rings

[beyond the "Drude formula"]

Low frequency absorption of metallic grains

[beyond Gorkov-Eliashberg]

Rate of heating

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$

$$g_s \equiv \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}$$

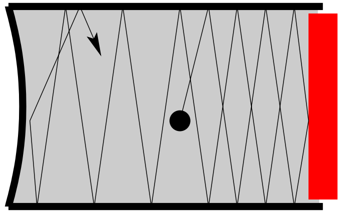
$f(t)$ = low freq noisy driving

$\leadsto$ diffusion in energy space:

$$D_{\text{SLRT}} = g_s \, D_{\text{LRT}}$$

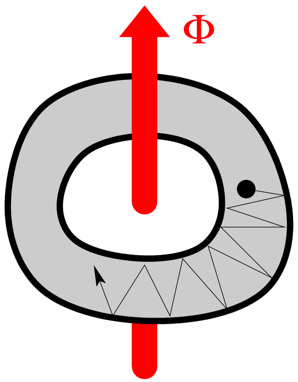
$\leadsto$ energy absorption:

$$\dot{\boldsymbol{E}} = (\text{particles/energy}) \times D$$



$$D_{\text{LRT}} = g_c \, \frac{4}{3\pi} \frac{\mathsf{m}^2 v_{\text{E}}^3}{L^2} L_{\text{wall}} \, \overline{\dot{f}^2}$$

[The "Wall Formula"]



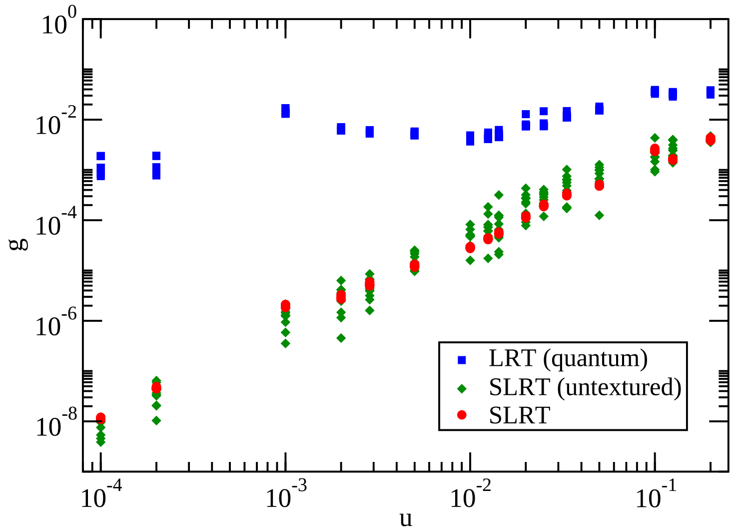
$$D_{\text{LRT}} = g_c \left(\frac{e}{L}\right)^2 v_{\text{E}} L_{\text{free}} \, \overline{\dot{\Phi}^2}$$

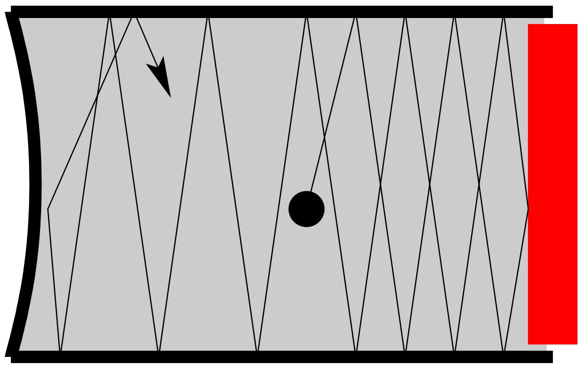
[The "Drude Formula"]

Some numerical results

Beyond the "Wall Formula"

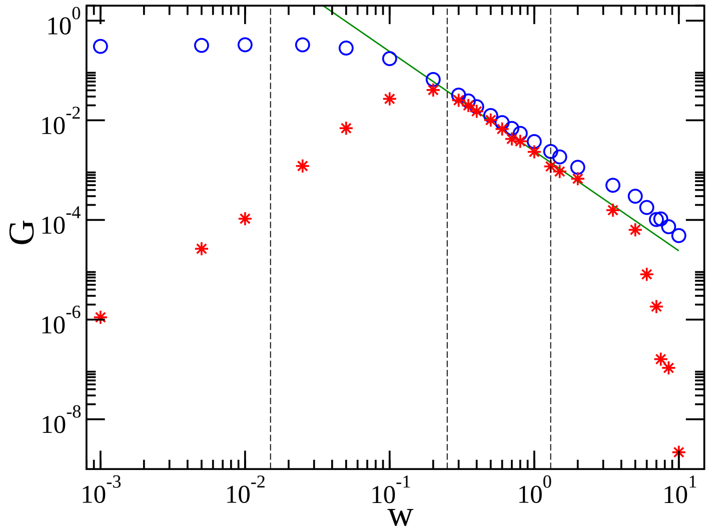
[Dependence on deformation U]

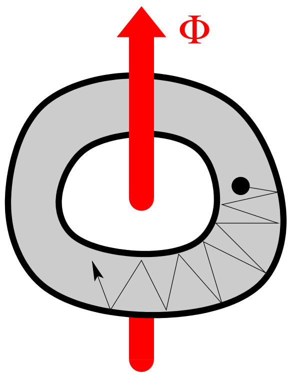




Beyond the "Drude Formula"

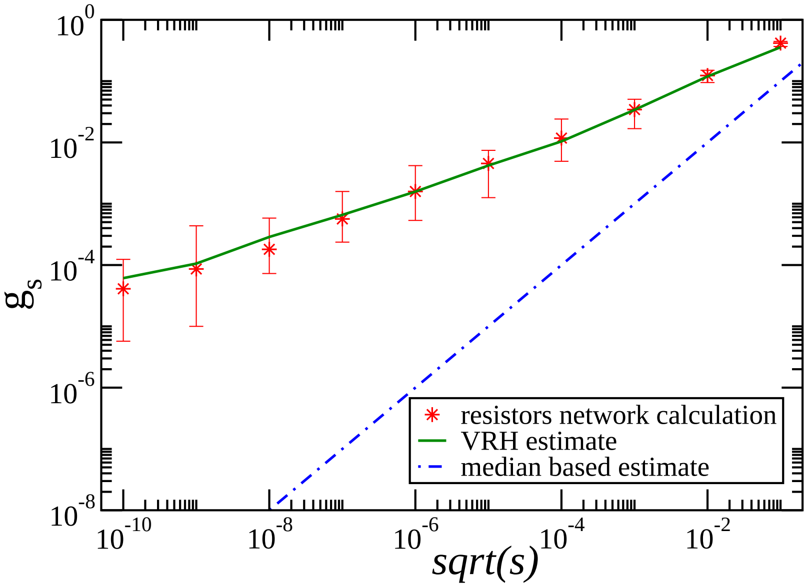
[Dependence on disorder W]





RMT modeling, generalized VRH approx scheme

- finite bandwidth b
- log-normal distribution s

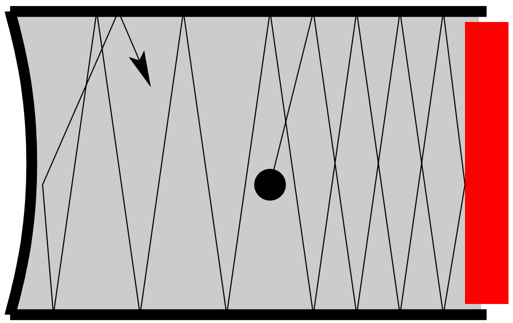


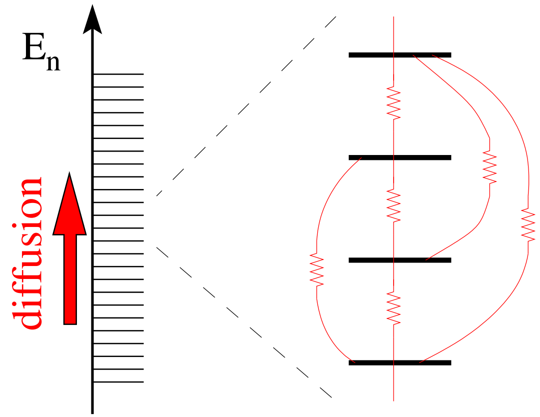
$$s \quad = \quad (\text{median}/\text{mean})^2$$

$$g_s \quad \approx \quad \sqrt{s}\ \exp\left[2\sqrt{-\ln\sqrt{s}\ \ln(b)}\right]$$

The energy absorption rate: the LRT to SLRT crossover

$$\mathcal{H}_{\text{total}} \;=\; \text{diag}\{E_n\} - f(t)V_{nm} + F(t)W_{nm} + Bath$$



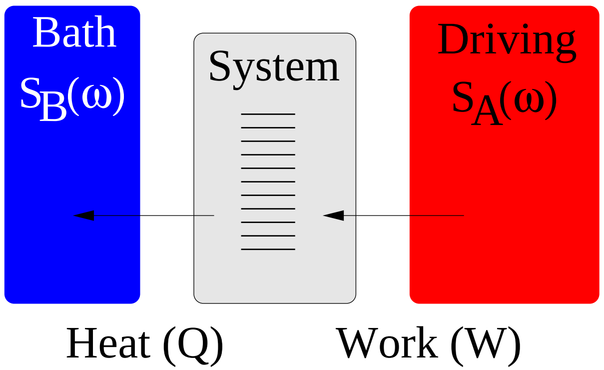


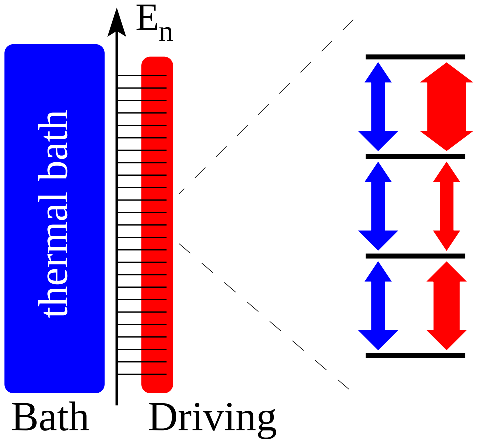
$\varepsilon^2 \equiv$ Driving intensity

$T_B \equiv$ Bath temperature

$$D_{LRT} \;=\; \overline{w_n}$$

$$D_{SLRT} \;=\; \left[\overline{1/w_n}\right]^{-1}$$





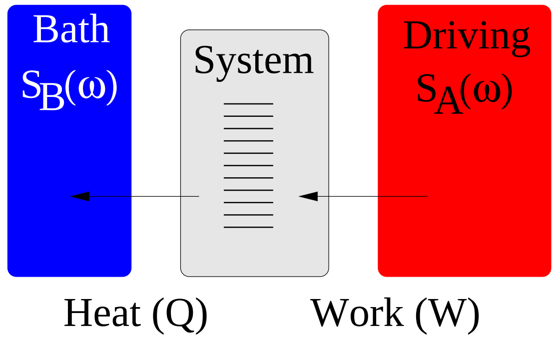
Expressions above assume near-neighbor transitions only.

The generalized Fluctuation-Dissipation phenomenology

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)V_{nm} + F(t)W_{nm} + \textit{Bath}$$

$$\dot{\mathsf{W}} = \text{rate of heating} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$\dot{\mathsf{Q}} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$



$D(\varepsilon)$ exhibits LRT to SLRT crossover

SLRT requires resistor network calculation

$$D_{LRT} = \overline{w_n}$$

$$D_{SLRT} = \left[\overline{1/w_n}\right]^{-1}$$

Hence at the NESS:

$$T_{\text{system}} = \left(1 + \frac{D(\varepsilon)}{D_B}\right) T_B$$

$$\dot{Q} = \dot{W} = \frac{1/T_B}{D_B^{-1} + D(\varepsilon)^{-1}}$$

Experimental way to extract response:

$$D(\varepsilon) = \frac{\dot{Q}(\varepsilon)}{\dot{Q}(\infty) - \dot{Q}(\varepsilon)} D_B$$

Semi-linear response:

$$D[\lambda \boldsymbol{w}] = \lambda D[\boldsymbol{w}]$$

$$D[\boldsymbol{w}^a + \boldsymbol{w}^b] > D[\boldsymbol{w}^a] + D[\boldsymbol{w}^b]$$

Experimental implication:

$$D\left[\lambda \tilde{S}(\omega)\right] = \lambda\, D\left[\tilde{S}(\omega)\right]$$

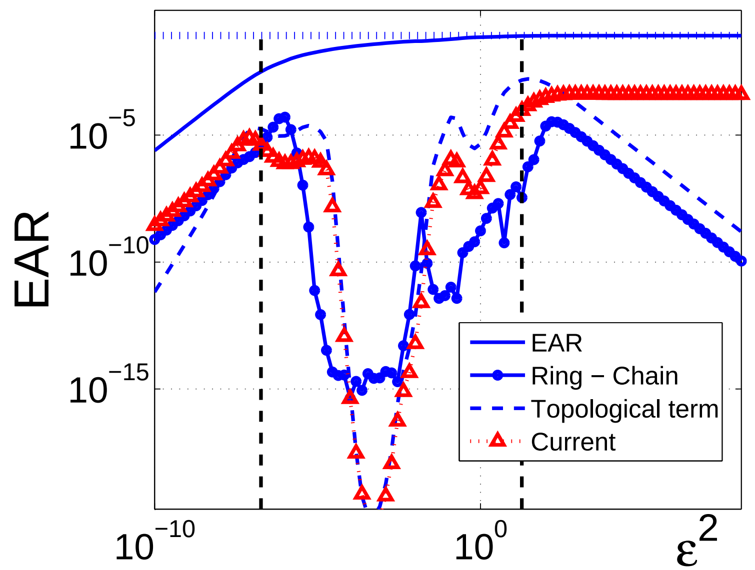
but...

Beyond the Fluctuation-Dissipation phenomenology: Topological term in EAR formula

$$\dot{Q} = \sum_n \left[w^{\beta}_{\overleftarrow{n}} p_n - w^{\beta}_{\overrightarrow{n}} p_{n-1} \right] \Delta_n$$

$$\approx \left[\frac{D_B}{T_B} - \frac{D_B}{T^{(0)}} \right] + T_B \mathcal{E}_{\circlearrowleft} I$$

$$\approx \frac{D_B}{T_B} \left[\overline{(g_n \epsilon^2) - (g_n \epsilon^2)^2} + \mathrm{Var}(g) \epsilon^4 \right]$$



The EAR is correlated with the current.

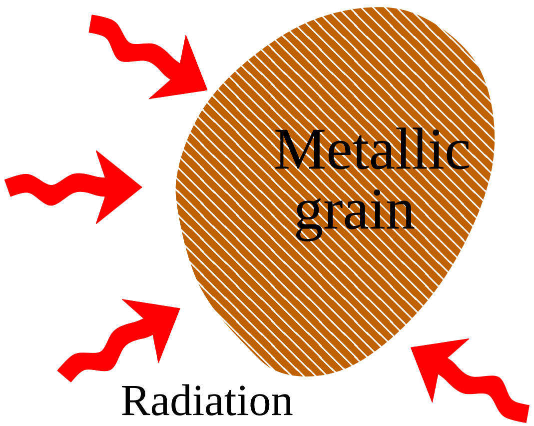
Why "sparsity" is interesting

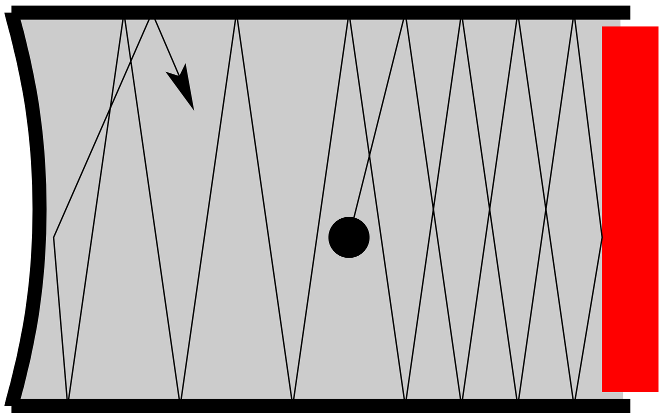
- The energy absorption rate (EAR) requires a resistor network calculation.
- Semi-linear response characteristics, beyond LRT.
- Sparsity implies a novel NESS that has glassy nature.
- Novel quantum saturation effect.
- Current versus driving: emergence of a Sinai regime.
- Applications:

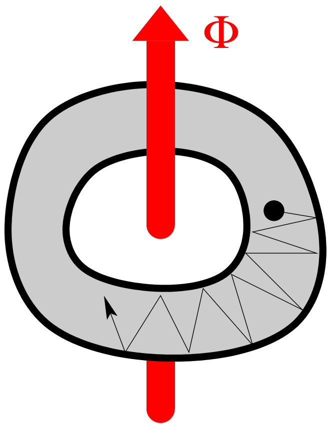
  Heating rate of cold atom in billiards with vibrating walls [beyond Swiatecki]

  Absorption of low frequency irradiation by metallic grains [beyond Gorkov-Eliashberg]

  Conductance of mesoscopic EMF-driven ballistic rings [beyond Drude]







Conclusions

(*) Wigner ($\sim$ 1955): "The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution." Not always...

1. "weak quantum chaos" $\Longrightarrow$ log-wide distribution, "sparsity" and "texture"
2. The heating $\sim$ a percolation process.
3. Resistors network calculation to get the response coefficient.
4. RMT modeling $\leadsto$ generalization of the VRH estimate.
5. Experimental fingerprint: semi-linear response characteristics.

6. SLRT applies if the driving is stronger then the background relaxation.
7. The stochastic NESS has glassy characteristics (wide distribution of microscopic temperatures).
8. Definition of effective NESS temperature, and extension of the F-D phenomenology.
9. For very strong driving - quantum saturation of the NESS temperature ($T \to T_{\infty}$).

10. Topological aspects: The emergence of the Sinai regime.
11. Topological term in the formula for the heating rate.

12. Applications: beyond the "Drude formula" and beyond the "Wall formula".

Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding g_c

Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding g_s

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

Non Equilibrium: Lebowitz, Spohn, Gaspard, Seifert, Schnakenberg, [...]

Sinai Physics: Sinai, Derrida, Pomeau, Burlatsky, Oshanin, Mogutov, Moreau, [...]

Acknowledgment: Bernard Derrida

[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)
[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)
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