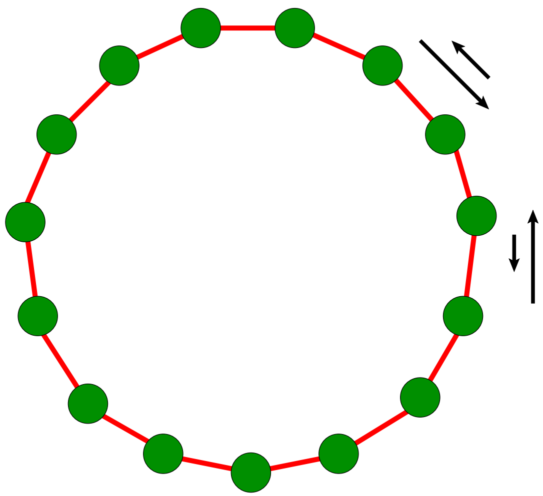
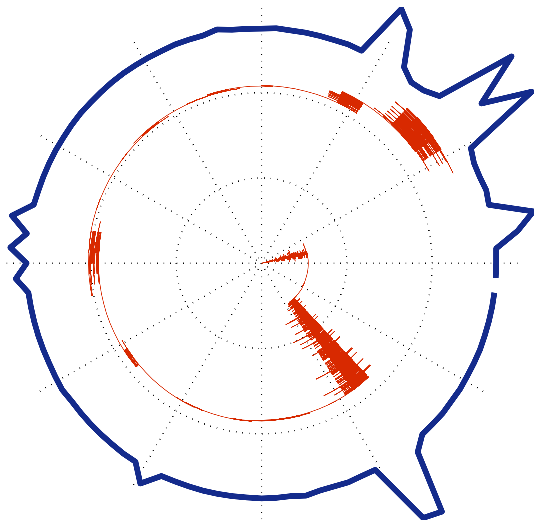
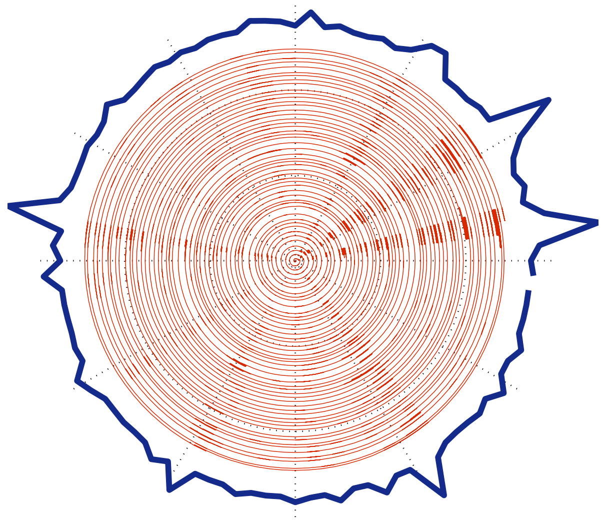
Percolation, sliding, localization and relaxation in topologically closed circuits

Doron Cohen

Ben-Gurion University







[**1**] Daniel Hurowitz, DC [**Scientific Reports 2016**]

[**2**] Daniel Hurowitz, DC [**PRE 2016**]

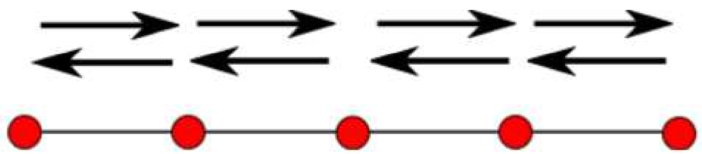
[**3**] Isaac Weinberg, Yaron de Leeuw, Tsampikos Kottos, DC [**PRE 2016**]

Types of random walk

Simple random walk, aka Brownian motion [Einstein]

Strictly periodic lattice ($a = 1$). All rates are equal (w)

$$D \;=\; w \quad \text{(near-neighbor hopping)}$$



Random walk on a disordered lattice [1]

Random lattice. Symmetric transition rates w_n

$$P(w) \;\propto\; w^{\alpha-1} \quad \text{(for small } w\text{)}$$

$\alpha \sim$ sparsity parameter

(resistor network calculation)

$$D \;=\; \left\langle \frac{1}{w} \right\rangle^{-1} \;=\; \text{conductivity}$$

Non-percolating for $\alpha < 1$

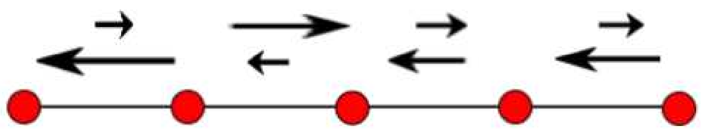
Percolation-like transition

Random walk in random environment [2]

Rates allowed to be asymmetric: $\overleftarrow{w}_n \neq \overrightarrow{w}_n$

Sub-diffusion for low bias [Sinai, Derrida,...]

Sliding transition



[1] Alexander, Bernasconi, Schneider, Orbach, Rev. Mod. Phys. (1981).

[2] Bouchaud, Comtet, Georges, Doussal, Annals Phys. (1990).

Definition of the model

Conservative rate equation

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{W}\boldsymbol{p}$$

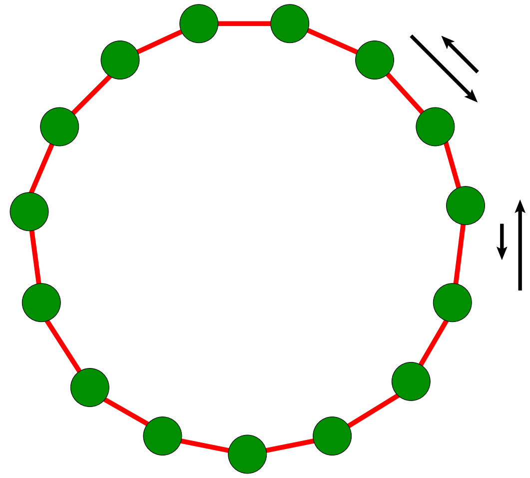
Rates allowed to be asymmetric $\overrightarrow{w}_n/\overleftarrow{w}_n = e^{\mathcal{E}_n}$

Affinity: $S_{\circlearrowleft} = \sum \mathcal{E}_n = Ns$

Stochastic field: $\mathcal{E}_n = s + \varsigma_n$ where $\varsigma_n \in [-\sigma, \sigma]$

Transition rates across n^{th} bond are $w_n \mathrm{e}^{\pm\mathcal{E}_n/2}$

Resistor network disorder: $P(w) \propto w^{\alpha-1}$



$$\boldsymbol{W} = \begin{bmatrix} -\gamma_1 & w_{1,2} & 0 & \dots \\ w_{2,1} & -\gamma_2 & w_{2,3} & \dots \\ 0 & w_{3,2} & -\gamma_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Sum of elements in each column is zero

How do spectral properties of $\boldsymbol{W}$ depend on (α, σ, s)?

$\alpha \sim$ sparasity, $\sigma \sim$ field disorder, $s \sim$ affinity

Related models

Vortex depinning in type II superconductors (s = applied transverse magnetic field)

- Hatano, Nelson, PRL (1996), PRB (1997).
- Shnerb, Nelson, PRL (1998).
- Follow ups: Brouwer, Silvestrov, Beenakker, PRB (1997). Goldsheid, Khoruzhenko, PRL (1998). Feinberg, Zee, PRE (1999). Molinari, Linear Algebra and its Applications (2008).

Pulling pinned polymers, DNA denaturation (s = pulling force)

- Lubensky, Nelson, PRL (2000), PRE (2002).

Population biology (s = convective flow of bacteria relative to the nutrients)

- Nelson, Shnerb, PRE (1998).
- Dahmen, Nelson, Shnerb, Springer (1999).

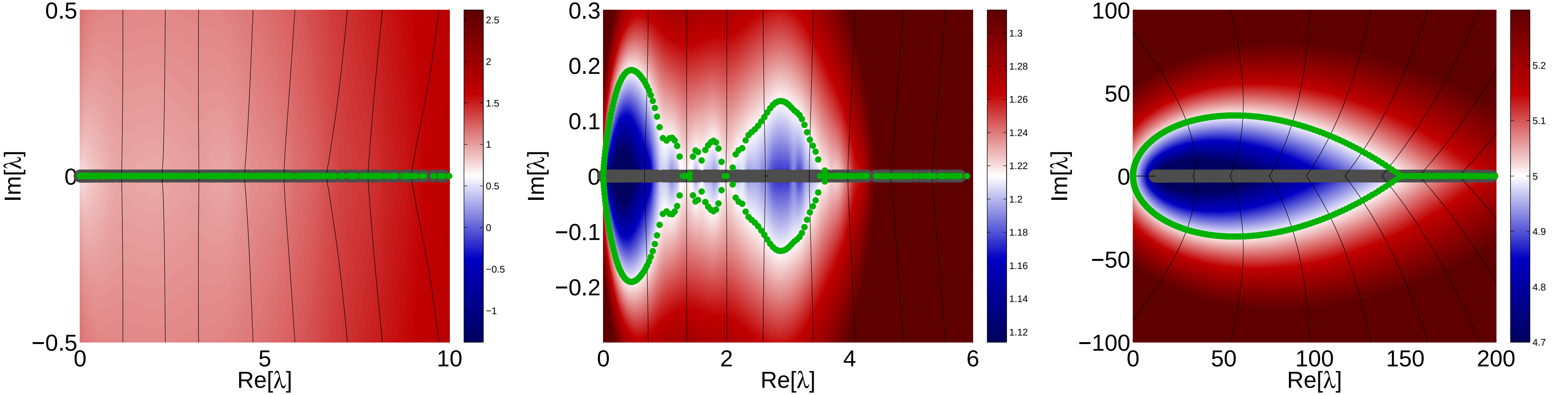
Molecular motors (s = affinity of chemical cycle)

- Fisher, Kolomeisky, PNAS (1999).
- Rief et al, PNAS (2000).
- Kafri, Lubensky, Nelson, Biophysical Journal (2004), PRE (2005).

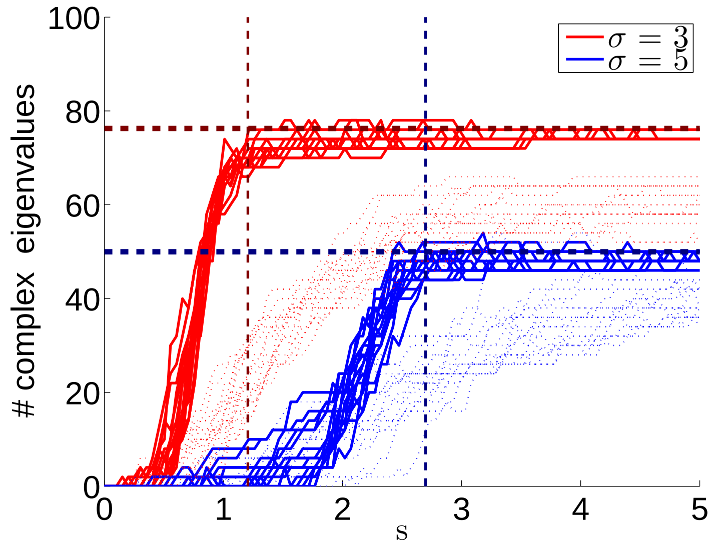
Non of the above concern relaxation modes of a conservative systems!

Implications of the percolation and sliding transitions on relaxation modes of the ring?

The spectrum of $\boldsymbol{W}$



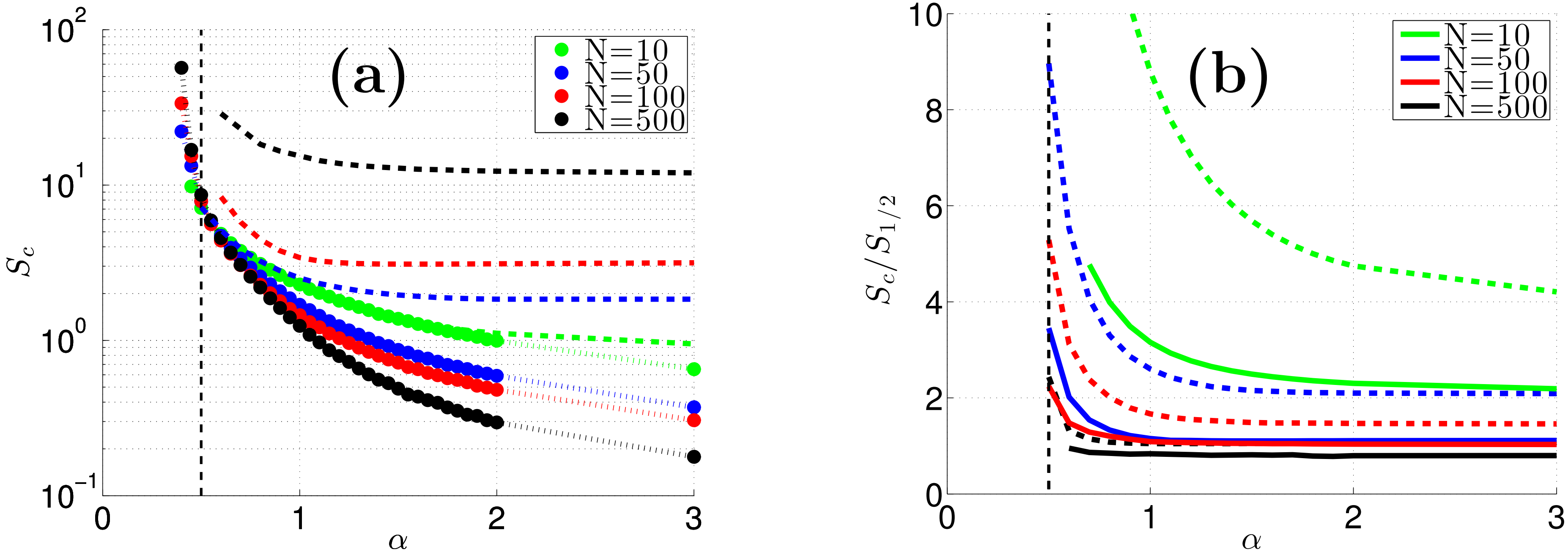
- Due to conservativity $\lambda_0 = 0$
- The other eigenvalues are $\{-\lambda_k\}$
- Complex bubble for $s > s_c$
- Complexity saturation for $s \gg s_\infty$
- Implication of the percolation transition?
- Implication of the sliding transition?



The complexity threshold (aka delocalization transition)

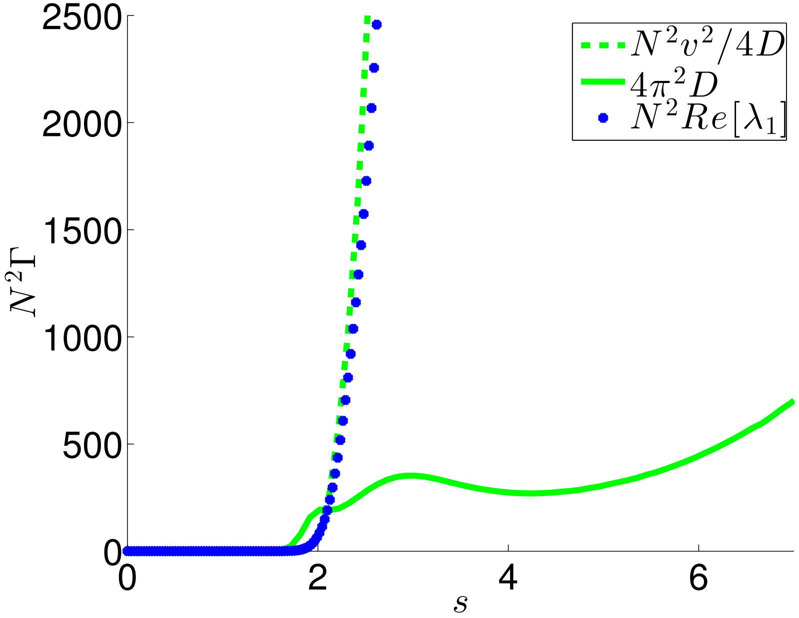
Naive expectation is $S_c = N\, s_1$ for $\alpha > 1$ and $S_c = \infty$ otherwise.

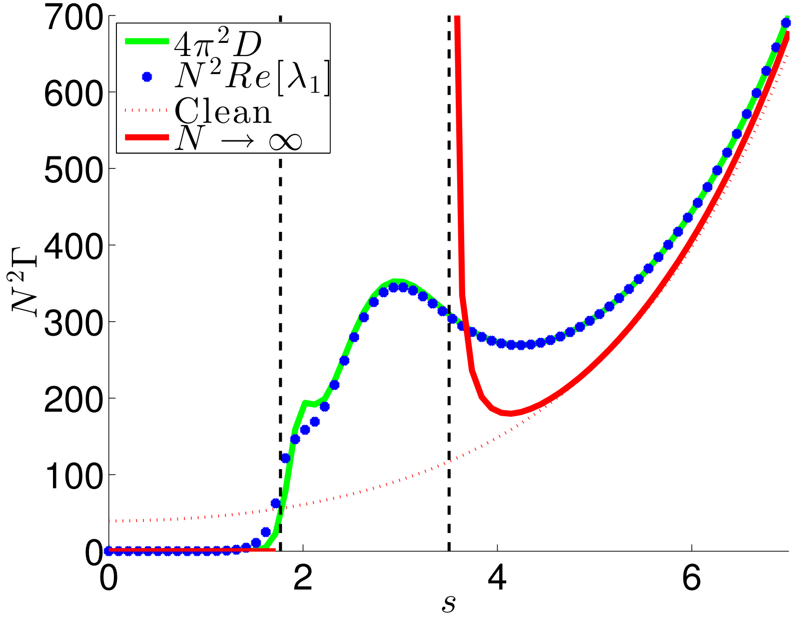
Type of disorder	Parameters		S_c for large N	Remarks
Resistor-network	$\alpha<\frac{1}{2}$	$\sigma=0$	$S_c = \infty$	non-percolating ("disconnected ring")
Resistor-network	$\frac{1}{2}<\alpha\ll 1$	$\sigma=0$	$S_c \sim \mathcal{O}(1)$	residual percolation ("weak link")
Resistor-network	$\alpha>1$	$\sigma=0$	$S_c \propto 1/\sqrt{N}$	percolating (conductivity $w_\infty>0$)
Stochastic field	$\alpha>\frac{1}{2}$	$\sigma>0$	$S_c \approx N\, s_{1/2}$	lower than sliding threshold at $N s_1$



The relaxation rate

Considering a ring of length $L = Na$ we define $\Gamma \equiv \mathrm{Re}[\lambda_1]$





$$\Gamma[\text{box}] = \left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{v}{2D}\right)^2\right] D$$

$$\Gamma[\text{ring}] = \left(\frac{2\pi}{L}\right)^2 D$$

Explicit results for D are known
in the $N \to \infty$ limit.
From the analysis of the spectrum we deduce:

$$\Gamma[\text{ring}] \propto \begin{cases} L^{-\frac{1}{\mu}} & \text{for } s_{1/2} < s < s_1 \\ L^{-\left(3-\frac{2}{\mu}\right)} & \text{for } s_1 < s < s_2 \\ L^{-2} & \text{for } s > s_2 \end{cases}$$

The characteristic equation

We are looking for the eigenvalues $\{-\lambda_k\}$ of the matrix $\boldsymbol{W}$.
The characteristic polynomial is:

$$\det(z+\boldsymbol{W}) \;=\; \det(z+\tilde{\boldsymbol{W}}) \;=\; \det(z+\boldsymbol{H}) \;-\; 2\left[\cosh\left(\frac{S_{\circlearrowleft}}{2}\right)-1\right]\prod_{n=1}^{N}(-w_n)$$

$$\boldsymbol{W} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\} + \text{offdiagonal}\Big\{w_n \mathrm{e}^{\pm\frac{\mathcal{E}_n}{2}}\Big\} \qquad \mathcal{E}_n \;=\; s+\varsigma_n$$

$$\tilde{\boldsymbol{W}} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\} + \text{offdiagonal}\Big\{w_n \mathrm{e}^{\pm\frac{S_{\circlearrowleft}}{2N}}\Big\} \qquad S_{\circlearrowleft} \;=\; N\,s$$

$$\boldsymbol{H} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\} + \text{offdiagonal}\Big\{w_n\Big\} \quad \rightsquigarrow \quad \text{diagonal}\Big\{-\epsilon_k(s)\Big\}$$

The characteristic equation:

$$\prod_k (z-\epsilon_k(s)) \;=\; 2\left[\cosh\left(\frac{N s}{2}\right)-1\right](-\overline{w})^N$$

The electrostatic version: $\quad \Psi(z)=\Psi(0)$

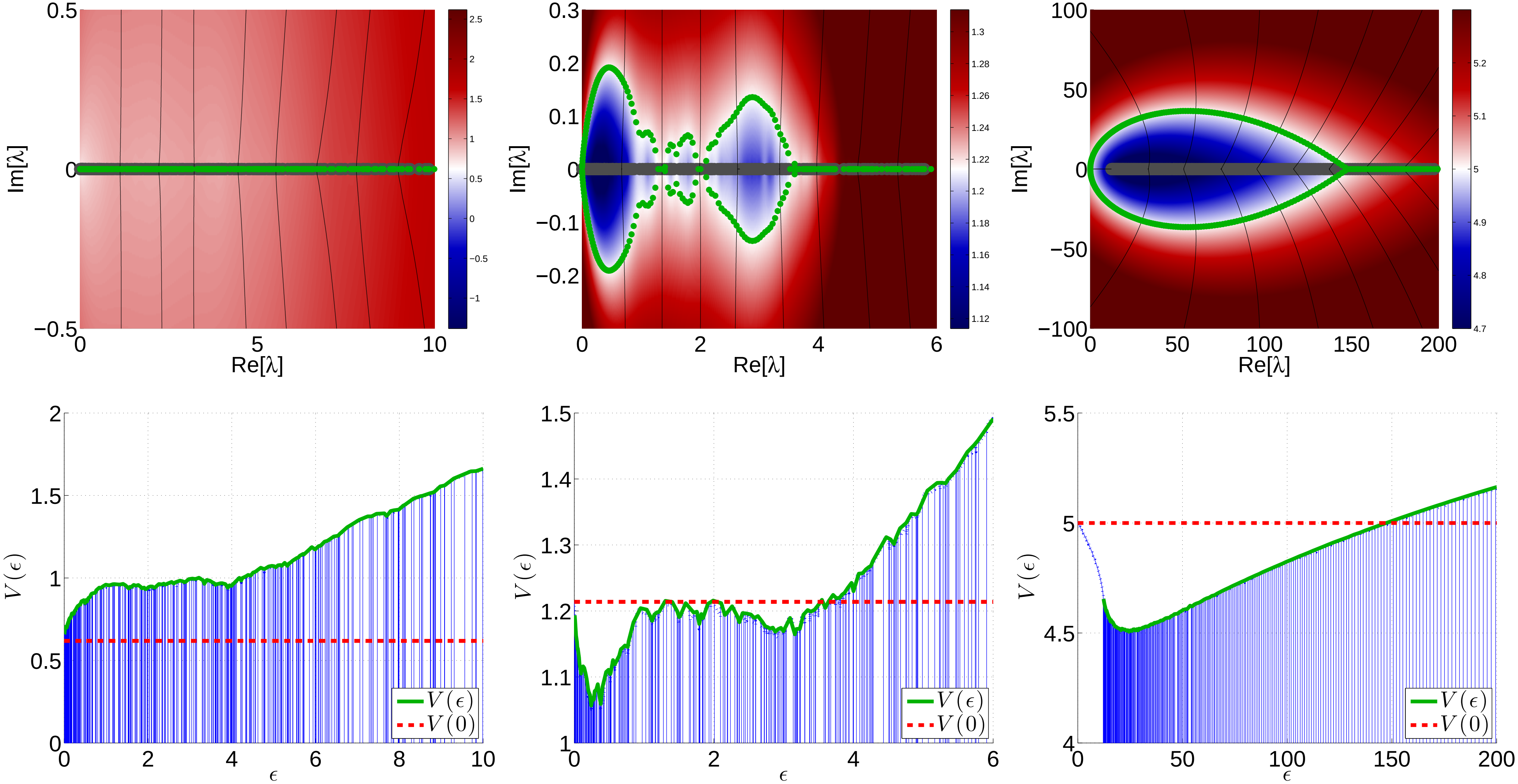
RHS is $\Psi(0)$ because $\lambda_0{=}0$ is in the spectrum.

$$\Psi(z) \;\equiv\; \sum_k \ln\left(z-\epsilon_k(s)\right) \;\equiv\; V(x,y)+iA(x,y)$$

The electrostatic picture

The complex potential: $\Psi(z) \;=\; \sum_k \ln(z-\epsilon_k) \;=\; V(x,y) + iA(x,y)$

Characteristic equation: $V(x,y) = V(0) \qquad A(x,y) = 2\pi * \text{integer}$



$s < s_c$ $\qquad s_c < s < s_\infty$ $\qquad s > s_\infty$

The formation of a complex bubble

The λ spectrum is real if $V(\epsilon) > V(0)$.
The characteristic equation is $V(\epsilon) = V(0)$
leading to $\lambda_k \sim \epsilon_k$

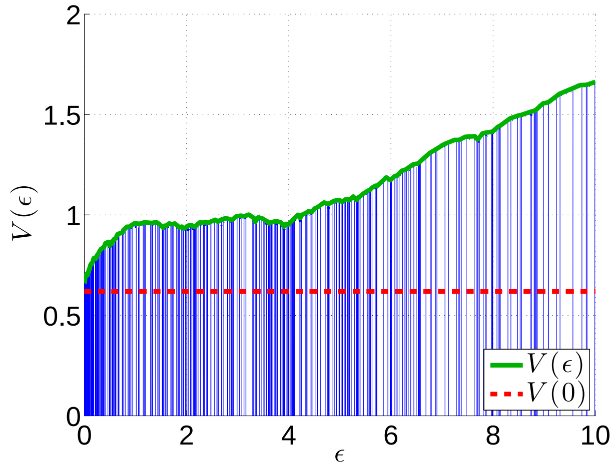
$$V(\epsilon) \quad = \quad \int \ln\left(\left|\epsilon - \epsilon'\right|\right)\rho(\epsilon')d\epsilon' \quad \sim \quad \text{inverse localization length}$$

- For Anderson problem - $V(\epsilon)$ diverges at the band edge
- For Debye model - $V(\epsilon)$ goes to zero at the band edge
- A conservative $\boldsymbol{H}$ is formally like Debye model
- As the affinity is increased the conservativity of $\boldsymbol{H}$ is spoiled.

Charge density:

$$\rho(\epsilon) \quad \equiv \quad \sum_k \delta(\epsilon - \epsilon_k(s))$$

Electrostatic potential:



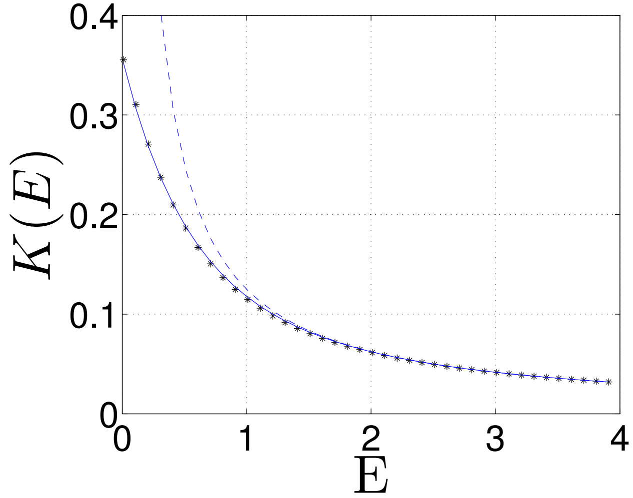
In non-conservative models [Hatano, Nelson, Shnerb]:

- The characteristic equation is $V(\epsilon) = f(s)$ with $f(s) \neq V(0)$
- A complex bubble appears in the middle of the spectrum, where $V(\epsilon)$ is small

Digression - Debye vs Anderson localization

The inverse localization length based on Halperin [PhysRev 1965] and Thouless relation:

$$V(\epsilon) \;=\; \left(\frac{\mathrm{Var}(v_n)}{2w_0^2}\right)^{1/3} \mathcal{K}\left[\left(2\frac{[\mathrm{Var}(v_n)]^2}{w_0}\right)^{-1/3} \epsilon\right]$$



$$\mathcal{K}(E) \;\approx\; \left[1 - \mathrm{e}^{-2.86E}\right]\frac{1}{8\,E}$$

Born approximation:

$$V(\epsilon) \;\approx\; \frac{1}{8}\left[\frac{\mathrm{Var}(v_n)}{w_0^2}\right]\left(\frac{\epsilon}{w_0}\right)^{-1}$$

We have established a duality that relates Debye localization to weak diagonal disorder:

$$V(\epsilon) \;=\; \left(\frac{1}{2}\left[w_\infty^2 \mathrm{Var}\left(\frac{1}{w_n}\right)\right]\left(\frac{\epsilon}{w_\infty}\right)^2\right)^{\frac{1}{3}} \mathcal{K}\left[\left(2\left[w_\infty^2 \mathrm{Var}\left(\frac{1}{w_n}\right)\right]^2\left(\frac{\epsilon}{w_\infty}\right)\right)^{-\frac{1}{3}}\right]$$

Dual Born approximation:

$$V(\epsilon) \;\approx\; \frac{1}{8}\left[w_\infty^2 \mathrm{Var}\left(\frac{1}{w_n}\right)\right]\left(\frac{\epsilon}{w_\infty}\right)$$

w_∞ = conductivity

The determination of s_c

In the absence of disorder we have a band. The band floor is

$$\epsilon_s \quad = \quad 2\left[\cosh(s/2) - 1\right]$$

For sparse disorder there are out-of-band impurities.

For full disorder if $s < s_\infty$ we have by def $\epsilon_s = 0$. Accordingly

$$\rho(\epsilon) \quad = \quad \epsilon^{\mu-1} \quad \text{(for small } \epsilon\text{)} \qquad [\mu \text{ depends on } s]$$

We define s_∞ as the value of s for which $\mu = \infty$

The threshold s_c is determined respectively from the conditions

$$V(\epsilon_s) < V(0) \qquad \text{or} \qquad V'(0) < 0$$

$$V(\epsilon) \quad = \quad \int \ln\left(\left|\epsilon - \epsilon'\right|\right)\rho(\epsilon')d\epsilon'$$

$$V'(\epsilon \to 0) \quad \approx \quad \frac{\epsilon^{\mu-1}}{\epsilon_c^{\mu}}\pi\mu\cot(\pi\mu)$$

The derivative changes sign from positive to negative at $\mu = 1/2$.

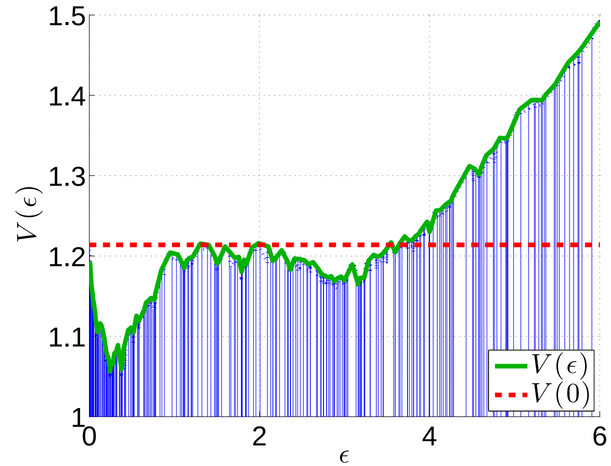
We define $s_{1/2}$ as the value of s for which $\mu = 1/2$

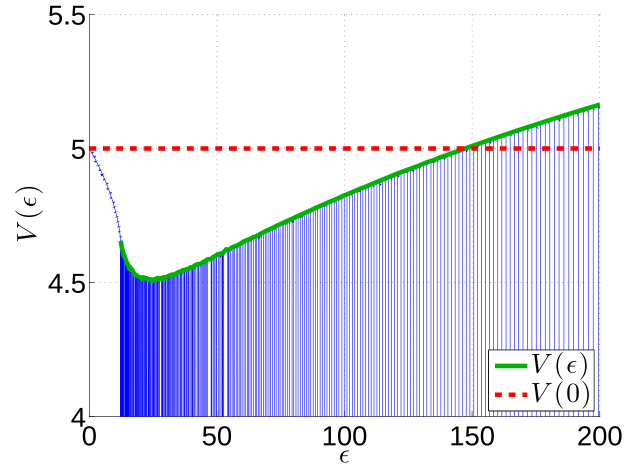
For full disorder we make the identification $s_c = s_{1/2}$.

Charge density:

$$\rho(\epsilon) \quad \equiv \quad \sum_k \delta(\epsilon - \epsilon_k(s))$$

Electrostatic potential:





Digression - the determination of μ

The thresholds s_μ are defined from

$$\left\langle e^{-\mu\mathcal{E}}\right\rangle \equiv \mathrm{e}^{-(s-s_\mu)\mu}$$

For an infinite chain:

$D=0$ for $s<s_{1/2}$,

$v=0$ for $s<s_1$.

For Gaussian disorder: $s_\mu = \frac{1}{2}\sigma^2\mu$

For Box disorder: $s_\mu = \frac{1}{\mu}\ln\left(\frac{\sinh(\sigma\mu)}{\sigma\mu}\right)$

In the latter case note that $s_\infty=\sigma$.

With a given s we associate μ such that $s=s_\mu$.

This μ is reflected in the time dependent spreading $x\sim t^\mu$

Correspondingly it is reflected in the density of eigenvalues:

$\rho(\epsilon) \propto \epsilon^{\mu-1}$ (for small ϵ)

For $s>s_\infty$ a gap is opened.

Resistor network disorder

Ientical bonds means $\alpha=\infty$.

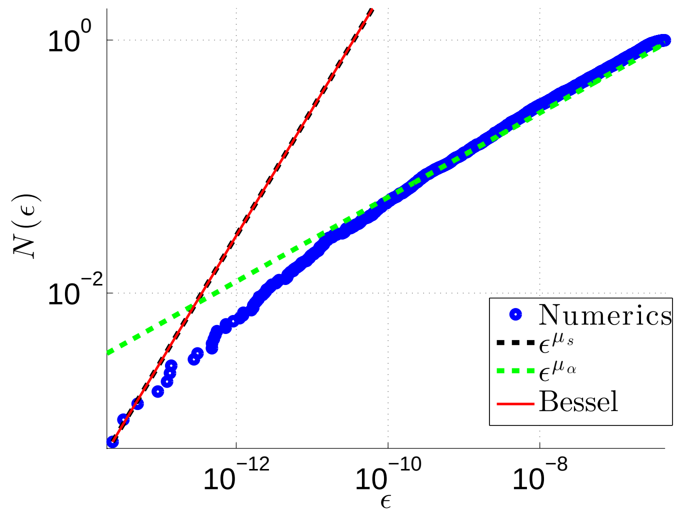
Otherwise we have $\mu(s)<\alpha<\infty$.

$\mu = \frac{\alpha}{1+\alpha}$ for $\alpha<1$, $s=\sigma=0$

$\mu = \frac{1}{2}$ for $\alpha>1$, $s=\sigma=0$

$\mu = \alpha$ adding large s

With field disorder ($\sigma>0$):



Digression - ring with a weak link

Using transfer matrix methods we find the characteristic equation in the continuum limit

$$\cos(k) - \frac{1}{g}\frac{k^2 + \left(\frac{S_{\circlearrowleft}}{2}\right)^2}{2k}\sin(k) \;=\; \cosh\left(\frac{S_{\circlearrowleft}}{2}\right)$$

$$k^2 \;=\; (L^2/D)z - (S_{\circlearrowleft}/2)^2$$

$$g \;=\; (D_1/a)/(D_0/L)$$

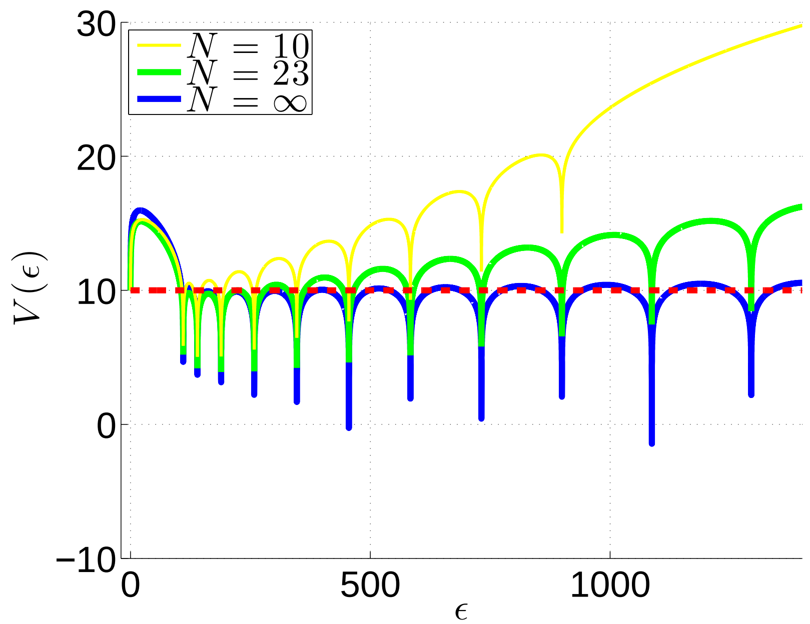
We have taken above the limit $a \to 0$, keeping $(L, g, S_{\circlearrowleft})$ constant.

The threshold S_c is the solution of

$$\frac{S_{\circlearrowleft}}{2g} \;=\; \cosh\left[\frac{S_{\circlearrowleft}}{2}\right]$$

We can identify the electrostatic potential as

$$V(\epsilon) \;=\; \ln[2(\mathrm{LHS}-1)] \;=\; \sum_{k=1}^{\infty}\ln|\epsilon - \epsilon_k|$$



Plot of the electrostatic potential along the real axis ($z = \epsilon$)

In order to reconstruct the potential from the ϵ_k we have to take into account the ϵ_0 that corresponds to an impurity-level associated with a mode which is located at the weak-link.

Complexity saturation

The cutoff frequency ϵ_c of the complex bubble is determined by the characteristic equation

$$\sum_k \ln|\epsilon - \epsilon_k(s)| \quad = \quad \ln\left\{2\left[\cosh\left(\frac{Ns}{2}\right) - 1\right]\right\}$$

We can solve it easily for $s \gg s_\infty$

$$\epsilon_k(s) \quad \mapsto \quad \gamma_n \quad \approx \quad w_n \mathrm{e}^{\mathcal{E}_n/2}$$

$$\epsilon_n(s) \quad = \quad w_n\, \mathrm{e}^{(s+\varsigma_n)/2} \qquad\qquad \varsigma_n \in [-\sigma, +\sigma]$$

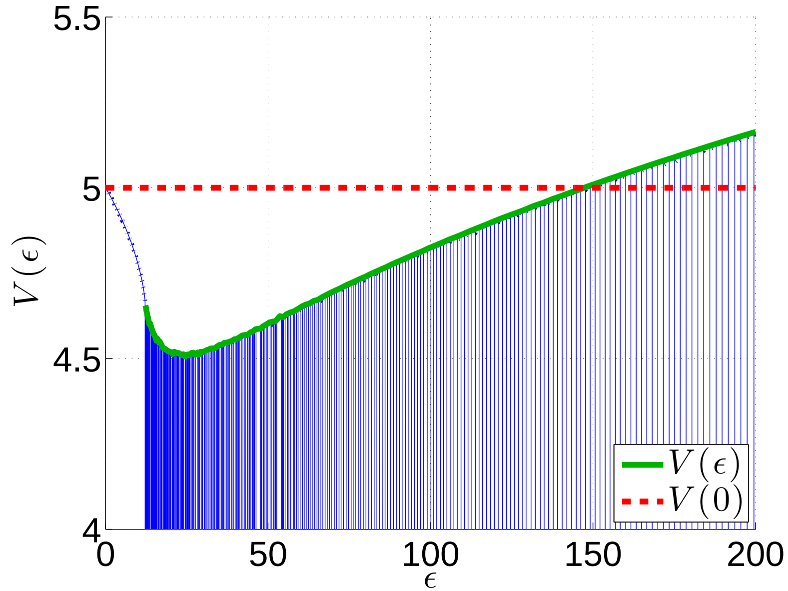
$$\epsilon_c(s) \quad \equiv \quad \bar{w} \quad \mathrm{e}^{(s+\sigma_c)/2}$$

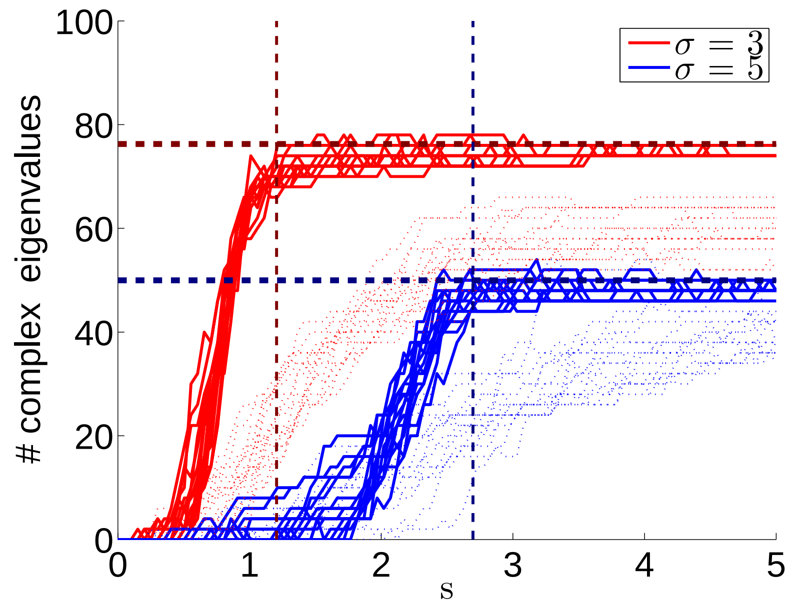
The characteristic equation takes the form

$$\frac{1}{N}\sum_n \ln\left|\bar{w}\mathrm{e}^{(s+\sigma_c)/2} - w_n \mathrm{e}^{(s+\varsigma_n)/2}\right| \quad = \quad \frac{s}{2}$$

We get an s independent equation for σ_c

$$\overline{\ln\left[\ \bar{w}\ \mathrm{e}^{\sigma_c/2} \ - \ w\ \mathrm{e}^{\varsigma/2}\ \right]} \quad = \quad 0$$





Summary

Relaxation properties of a closed circuit, whose dynamics is generated by a conservative rate-equation, are dramatically different from that of a biased non-hermitian Hamiltonian.

Type of disorder	Parameters		S_c for large N	Remarks
Resistor-network	$\alpha < \frac{1}{2}$	$\sigma = 0$	$S_c = \infty$	non-percolating ("disconnected ring")
Resistor-network	$\frac{1}{2} < \alpha \ll 1$	$\sigma = 0$	$S_c \sim \mathcal{O}(1)$	residual percolation ("weak link")
Resistor-network	$\alpha > 1$	$\sigma = 0$	$S_c \propto 1/\sqrt{N}$	percolating (conductivity $w_{\infty} > 0$)
Stochastic field	$\alpha > \frac{1}{2}$	$\sigma > 0$	$S_c \approx N\, s_{1/2}$	lower than sliding threshold at $N s_1$

- Relaxation becomes under-damped due to the appearance of a complex-bubble at the band floor.
- Resistor network disorder - Transition to complexity happens **before** the percolation transition.
- Stochastic field disorder - Transition to complexity happens **before** the sliding transition.
- Increasing further the affinity - **"complexity saturation"** is observed.