Celebrating 70th birthday of Amnon Aharony

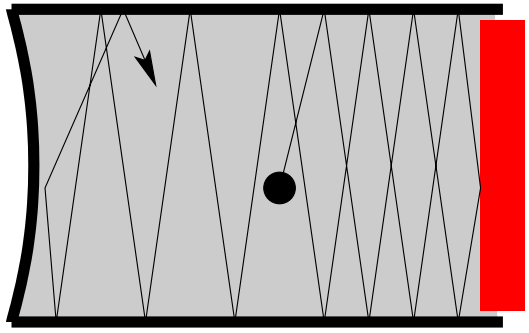
Response characteristics of sparse (glassy) networks

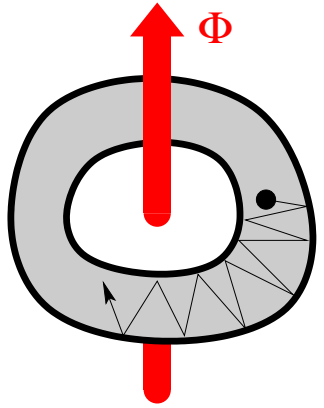
Doron Cohen

Ben-Gurion University

$$\frac{d\boldsymbol{p}}{dt} = \left[\mathcal{W}^{\beta} + \mathcal{W}\right]\,\boldsymbol{p} \qquad\qquad w_{nm}^{\text{total}} = w_{nm}^{\beta} + \nu g_{nm}$$

Yaron de Leeuw (BGU) [1]
Daniel Hurowitz (BGU) [2,3]
Saar Rahav (Technion) [3]
Alex Stotland (BGU) [former works]





[1] Y. de Leeuw, D. Cohen, PRE (2012).
[2] D. Hurowitz, D. Cohen, EPL (2011).
[3] D. Hurowitz, S. Rahav, D. Cohen, EPL (2012).

http://www.bgu.ac.il/∼dcohen

$DIP, $BSF, $ISF

"Sparsity" in oil reservoirs

Physica A 177 (1991) 260–266
North-Holland

Effective renormalization group algorithm for transport in oil reservoirs

Amnon Aharony [a,b], Einar L. Hinrichsen [b], Alex Hansen [b], Jens Feder [b], Torstein Jøssang [b] and H.H. Hardy [c]

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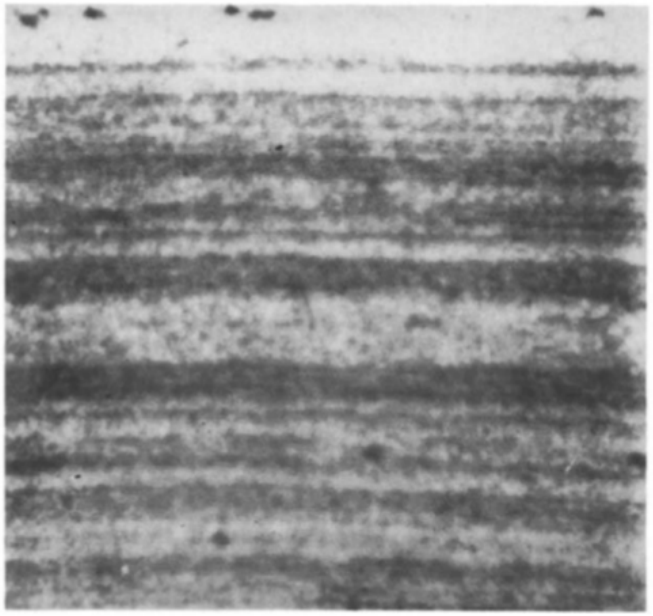
[c] *Conoco Inc., PER Department, Production R & D Division, Ponca City, OK 74603, USA*

Dedicated to Michael E. Fisher on the occasion of his 60th birthday



Each bond (=tube=pixel) is characterized by its local "permeability" $\kappa(\phi)$.

The issue is to calculate the macroscopic permeability $\overline{K}$ of the network.

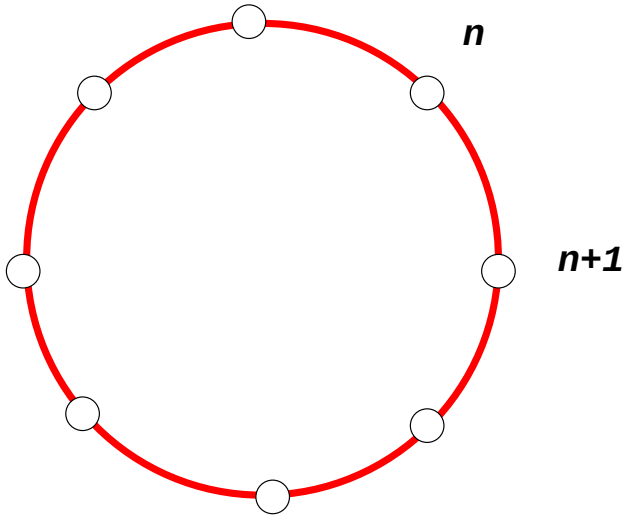
$$\kappa = 10^{A\phi + B} .$$

Since the pixel values have a more or less Gaussian distribution, the permeabilities can be made to have a *logarithmically broad distribution* by using a large value of A.

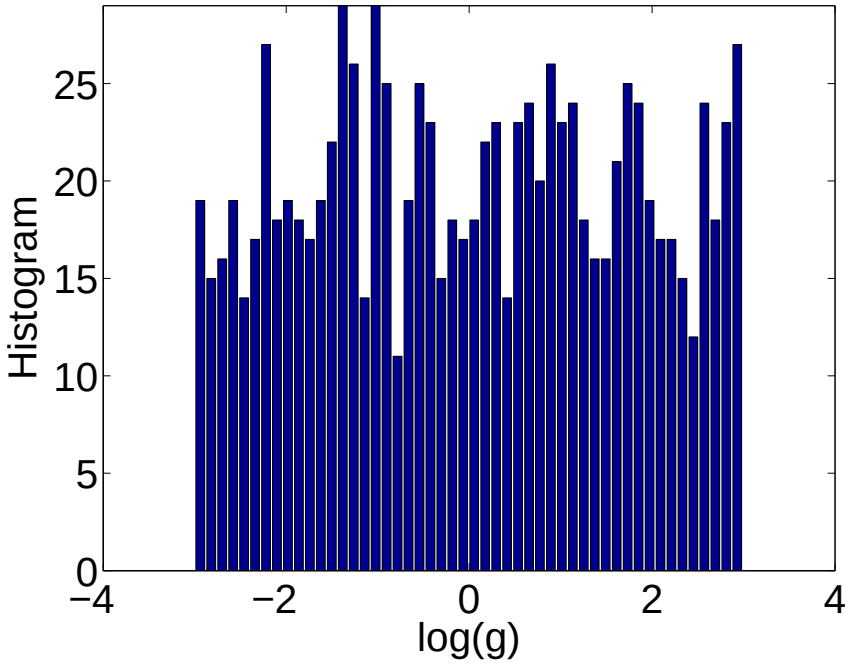
"Sparsity" of a constructed toy model

$$\frac{d\boldsymbol{p}}{dt} = \left[\mathcal{W}^{\beta} + \mathcal{W}\right] \boldsymbol{p}$$

$$w_{n+1,n} = g_n \nu$$



Histogram of the couplings



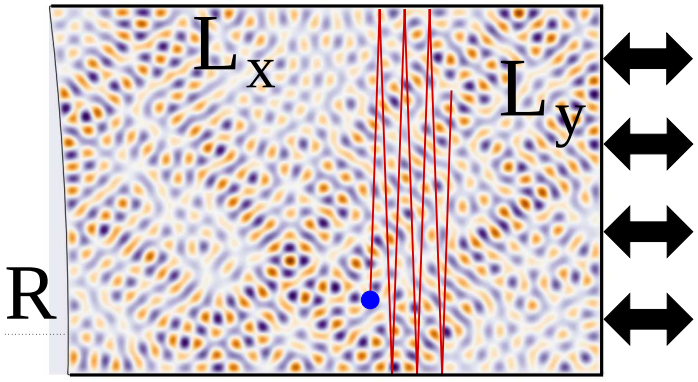
$\longleftarrow$ few decades $\longrightarrow$

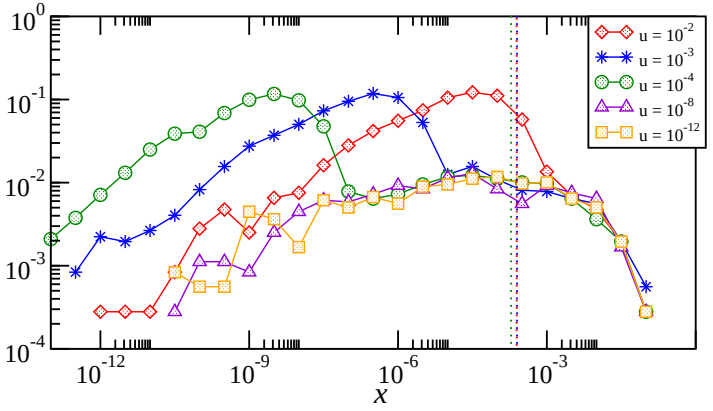
"sparsity" = log wide distribution of couplings

[**Hurowitz**, Rahav, Cohen (2012)]

"Sparsity" of a weakly chaotic driven system

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)V_{nm}$$





[log-wide distribution]

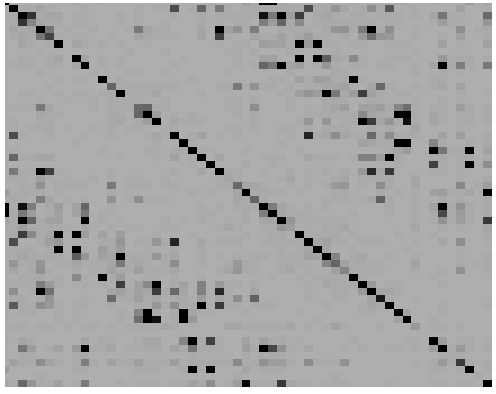
Stotland, Budoyo, Peer, Kottos, Cohen (2008),

Stotland, Cohen, Davidson (2009),

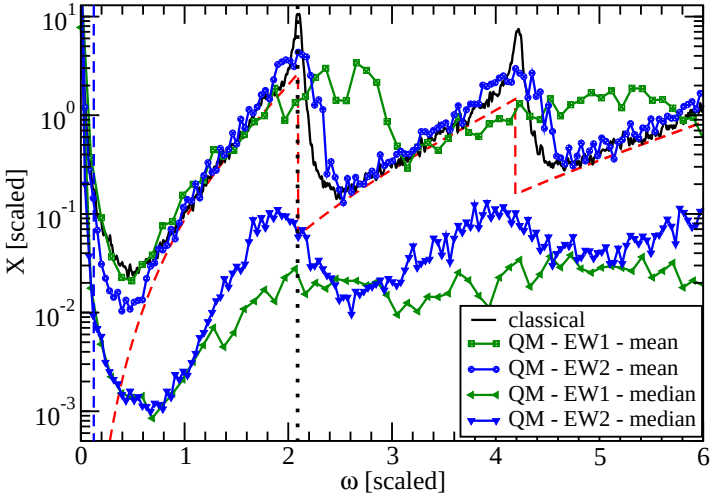
Stotland, Kottos Cohen (2010),

Stotland, Pecora, Cohen (2010,2011)

$$w_{n,m} = \nu\,|V_{nm}|^2$$



[not a Gaussian matrix...]



[median ≪ mean]

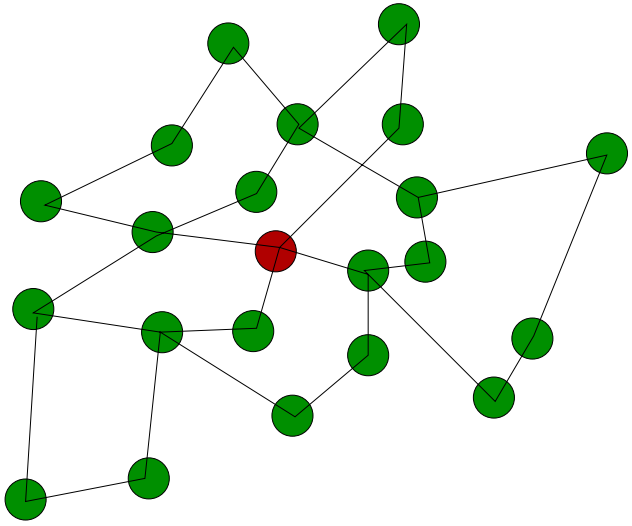
"Sparsity" in the Mott hopping model

$$w_{nm} = w_0 \, \mathrm{e}^{-\epsilon_{nm}} \, \exp\left[-\frac{r_{nm}}{\xi}\right]$$

$$\rho(r,\epsilon)drd\epsilon = \frac{\Omega_d \, r^{d-1} dr}{r_0^d} \, f(\epsilon)d\epsilon, \qquad \Omega_d = 2, 2\pi, 4\pi$$

$f(\epsilon) = 1$ Mott hopping model

$f(\epsilon) = \delta(\epsilon)$ Degenerate hopping model



Definition of "sparsity" parameter:

$$s \equiv \frac{\xi}{r_0}$$

$s \ll 1$ means "sparse"

For the non-degenerate Mott hopping model:

$$r_0^d = \left(\frac{\Delta_\xi}{T}\right) \xi^d \qquad \rightsquigarrow \qquad s = \left(\frac{T}{\Delta_\xi}\right)^{1/d}$$

[Miller, Abrahams (1960), Ambegaokar, Halperin, Langer (1971), Pollak (1972), Aharony et al (1991),..., Amir, Oreg, Imry (2010), **de Leeuw**, Cohen (2012), Amir, Krich, Vitelli, Oreg, Imry (arXiv)]

The definition of "sparse" random network

Random site model (1D/2D): sparsity affected by ξ

$$w_{nm} = w_0 \, \mathrm{e}^{-\epsilon_{nm}} \, \mathrm{e}^{-|x_n - x_m|/\xi}$$

x_n = random locations

ϵ_{nm} = random couplings or random activation potential

Banded lattice model (quasi-1D): sparsity affected by b

$$w_{nm} = w_0 \, \mathrm{e}^{-\epsilon_{nm}} \, B\left(E_n - E_m\right)$$

The models are characterized by two functions:

$w(r, \epsilon)$ = dependence of the rates on r and ϵ

$\rho(r, \epsilon)$ = distribution of "distances" in (r, ϵ) space

Remark: For a 1D chain with n.n. transitions, random r is like random ϵ,

resulting in rates that have the distribution [Alexander, Bernasconi, Schneider, Orbach (1981)]

$$f(w) \propto w^{s-1}, \quad (w < w_0), \qquad\qquad s \equiv \xi / r_0$$

The determination of D

$\boldsymbol{w}$ = symmetric matrix that represents a "sparse" network

$D \quad = \quad [[w]]$ (resistor network calculation)

Can be determined via numerical solution of a circuit equation $\boldsymbol{GV} = \boldsymbol{I}$

D is reflected in:

$$S(t) \quad = \quad \left\langle r^2(t) \right\rangle \quad \sim \quad Dt \qquad \text{long time spreading}$$

$$\mathcal{P}(t) \quad \sim \quad \frac{1}{(Dt)^{d/2}} \qquad \text{long time survival probability}$$

$$\mathcal{N}(\lambda) \quad \sim \quad \left[\frac{\lambda}{D}\right]^{d/2} \qquad \text{spectral function}$$

Can be determined via numerical diagonalization, small λ asymptotics

Random site model, the 1D case

The resistor network calculation can be done analytically for a chain with near neighbor transitions [Alexander, Bernasconi, Schneider, Orbach (1981)]

$f(w) \propto w^{s-1}, \quad (w < w_0), \qquad\qquad s \equiv \xi / r_0$

For $s > 1$ we get diffusion:

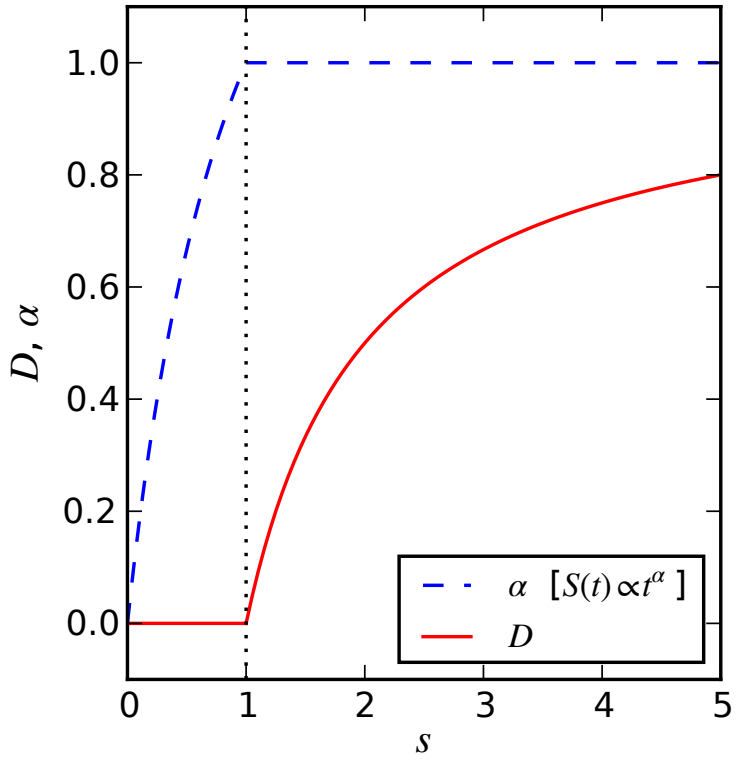
$$D \;=\; \left(\frac{1}{N} \sum_n \frac{1}{w_n} \right)^{-1} \;=\; \frac{s-1}{s}\, w_0,$$

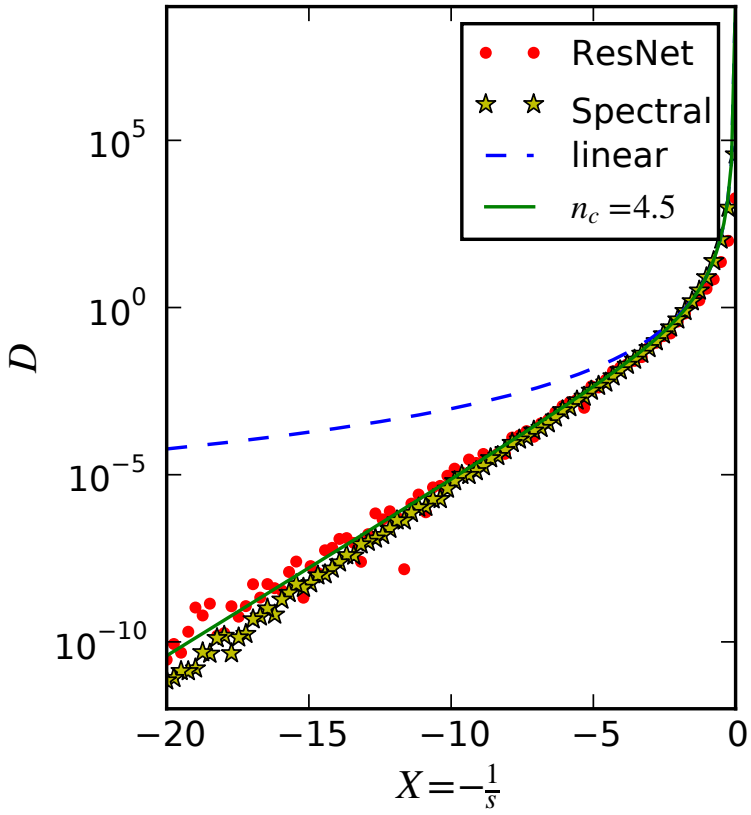
For $0 < s < 1$ we get sub-diffusion ($D = 0$) with:

$$S(t) \sim t^{2s/(1+s)}, \qquad \mathcal{P}(t) \sim t^{-s/(1+s)}, \qquad \mathcal{N}(\lambda) \sim \lambda^{s/(1+s)}$$

This is reflected in the spectral analysis.

Random site model, 2D case vs 1D case

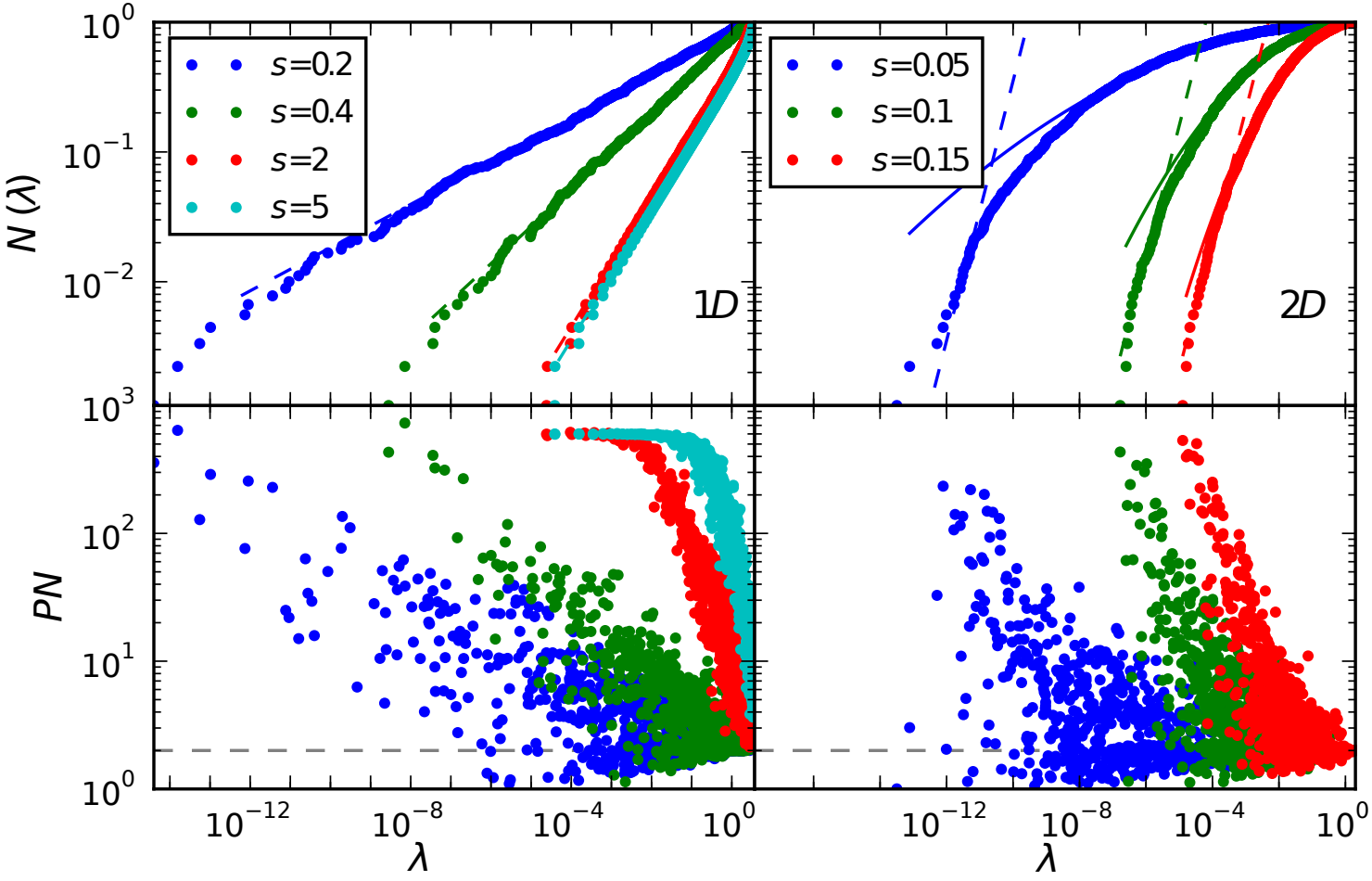




$$D_{\mathrm{ERH}} \quad = \quad \mathrm{EXP}_{d+2}\left(\frac{1}{s_{eff}}\right)\ \mathrm{e}^{-1/s_{eff}}\ D_{\mathrm{linear}} \qquad\qquad s_{eff} = \left(\frac{d}{\Omega_d} n_c\right)^{-1/d} \frac{\xi}{r_0}$$

For the non-degenerate model s_{eff} depends on the temperature, and use EXP_{d+3}

Spectral signature: 2D vs 1D



Solid lines - RG theory prediction [**Amir, Oreg, Imry (2010)**].

Dashed line - Resistor network theory prediction [**YdL, DC (2012)**].

Current status regarding the localization issue

Localization in a disordered elastic medium near two dimensions

Sajeev John, H. Sompolinsky, M.J. Stephen, Phys. Rev. B 27, 55925603 **(1983)**

- All states are localized
- Localization length diverges for $\lambda \to 0$
- Debye law holds ($\leftrightarrow$ diffusion)

Localization, Anomalous Diffusion, and Slow Relaxations

A. Amir, Y. Oreg, Y. Imry, Phys. Rev. Lett. 105, 070601 **(2010)**

- All states are localized
- There is sub-diffusion rather than difussion

Diffusion in sparse networks: Linear to semilinear crossover

Y. de Leeuw, D. Cohen, Phys. Rev. E 86, 051120 **(2012)**

- All states are localized
- There is a percolation crossover at finite λ
- Localization length diverges for $\lambda \to 0$
- Debye law holds ($\leftrightarrow$ diffusion)

Emergent percolation length and localization in random elastic networks

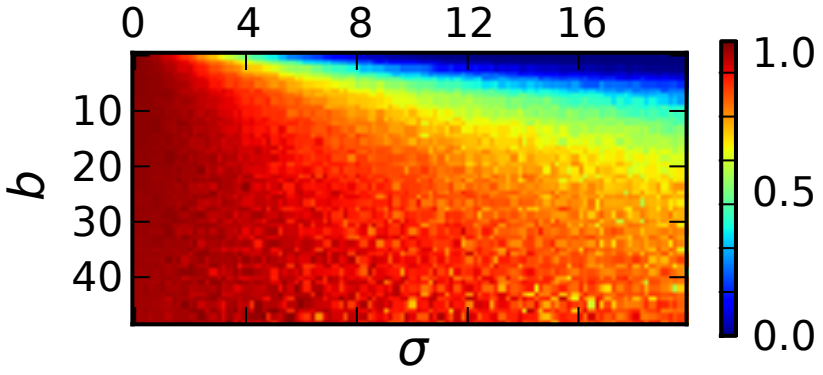
A. Amir, J. J. Krich, V. Vitelli, Y. Oreg, Y. Imry, **arXiv**

- There is an Anderson mobility edge at finite λ
- The low lying states become de-localized
- Debye law holds ($\leftrightarrow$ diffusion)

Linear and ERH estimates

$$D_{\text{linear}} = \frac{1}{2d}\int\int w(r,\epsilon) r^2 \rho(r,\epsilon) d\epsilon dr$$

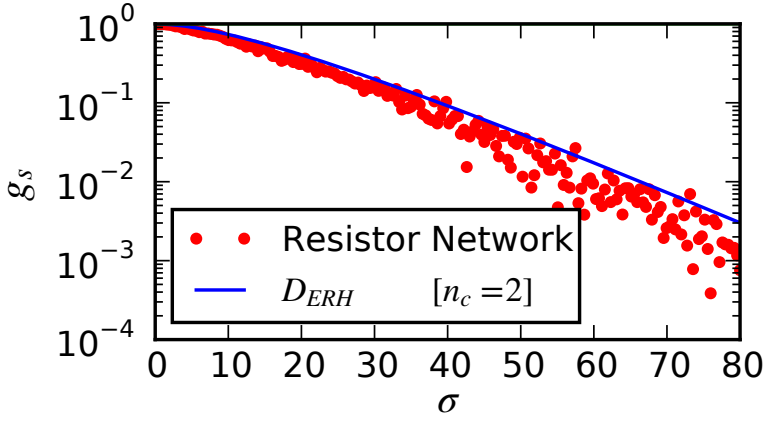
$$g_s = D/D_{linear}$$



This will not work for "sparse" systems:
only percolating paths contribute to the transport.
We define a percolation threshold by:

$$\int\int_{w(r,\epsilon)>w_c} \rho(r,\epsilon) dr d\epsilon = n_c$$

$n_c = 2$ for 1D networks
$n_c \approx 4.2$ for the random site model in 2D



Effective range hopping (ERH) estimate:

$$D_{\text{ERH}} = \frac{1}{2d}\int\int \min\{w(r,\epsilon), w_c\} r^2 \rho(r,\epsilon) d\epsilon dr$$

ERH vs VRH

ERH threshold w_c is defined via

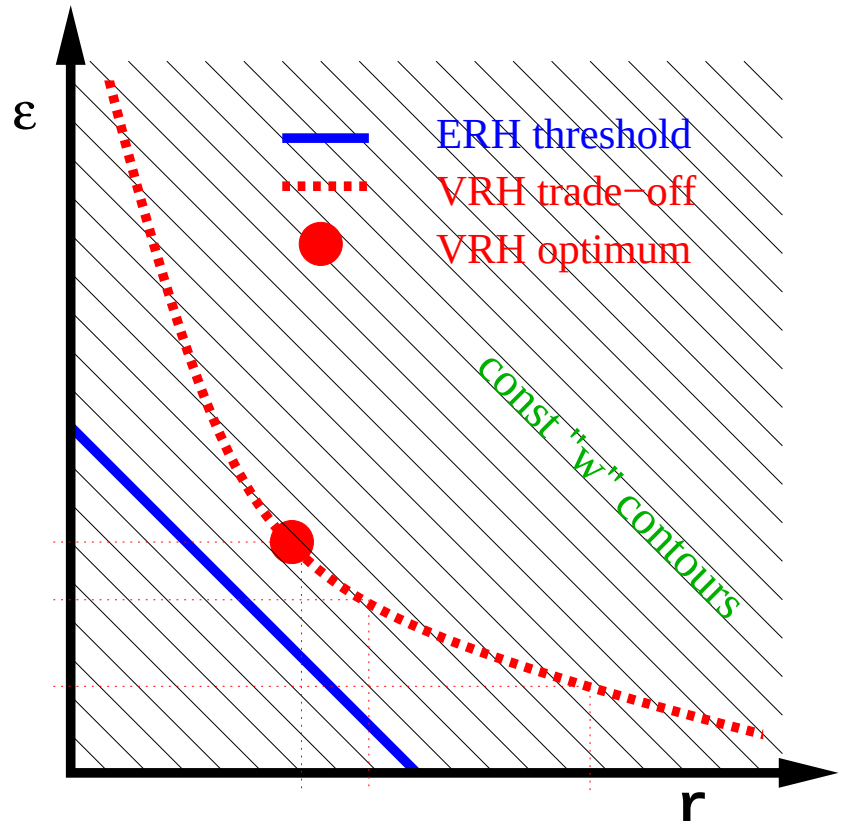
$$\int\int_{w(r,\epsilon)>w_c} \rho(r,\epsilon)drd\epsilon \quad = \quad n_c$$

VRH trade-off is along

$$\int_0^{\epsilon}\int_0^{r} \rho(r',\epsilon')dr'd\epsilon' \quad = \quad n^*$$

VRH optimum is the (r^*, ϵ^*) point
where $w(r,\epsilon)$ is maximal along this curve.

VRH estimate is $\quad D \ \sim \ w(r^*,\epsilon^*) \times (r^*)^2$



consistency if $n^* = n_c/d$

Application: rate of heating

$$\mathcal{H}_{\rm total} = \{E_n\} - f(t)\{V_{nm}\}$$

$$g_s \equiv \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}$$

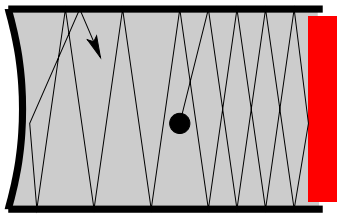
$f(t)$ = low freq noisy driving

$\rightsquigarrow$ diffusion in energy space:

$$D_{\rm SLRT} = g_s\, D_{\rm LRT}$$

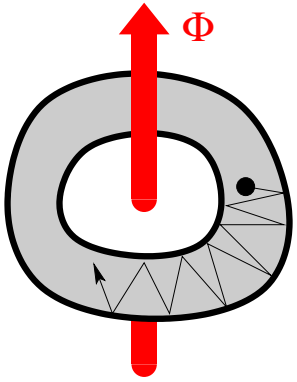
$\rightsquigarrow$ energy absorption:

$$\dot{E} = (\text{particles/energy}) \times D$$



$$D_{\rm LRT} = g_c \frac{4}{3\pi} \frac{\mathsf{m}^2 v_{\rm E}^3}{L^2} L_{\rm wall}\ \overline{\dot{f}^2}$$

[The "Wall Formula"]



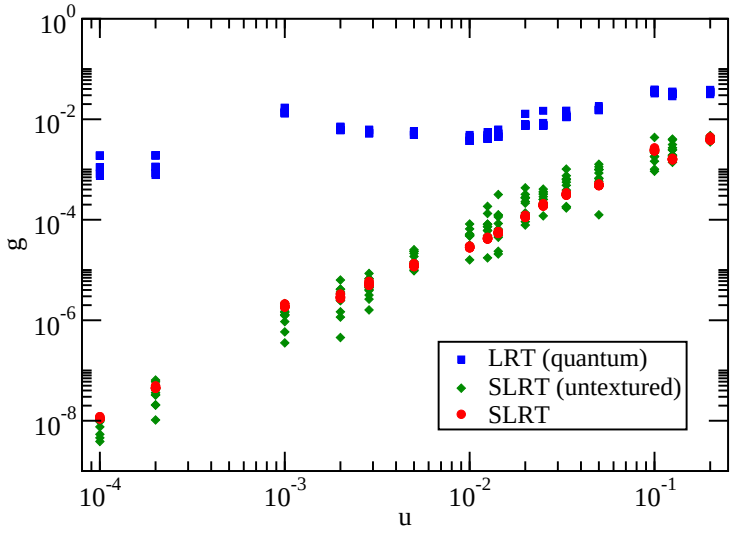
$$D_{\rm LRT} = g_c \left(\frac{e}{L}\right)^2 v_{\rm E} L_{\rm free}\ \overline{\dot{\Phi}^2}$$

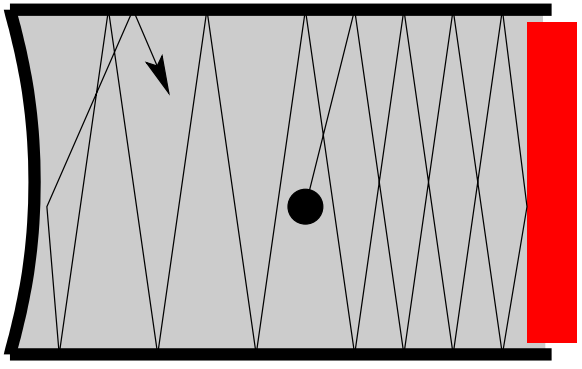
[The "Drude Formula"]

Some numerical results

Beyond the "Wall Formula"

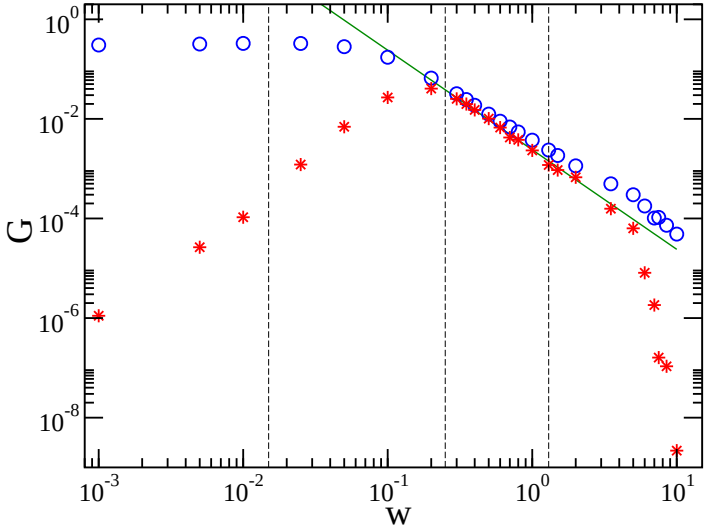
[Dependence on deformation U]

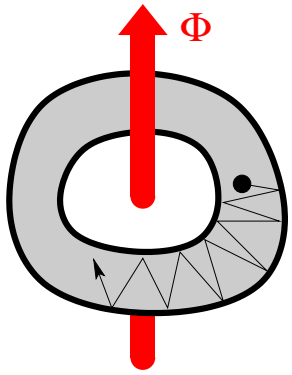




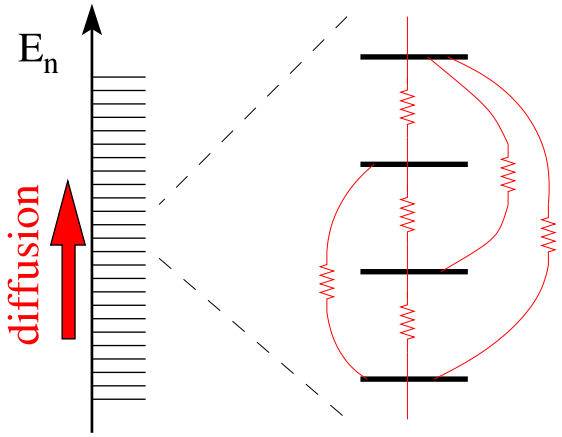
Beyond the "Drude Formula"

[Dependence on disorder W]





The energy absorption rate

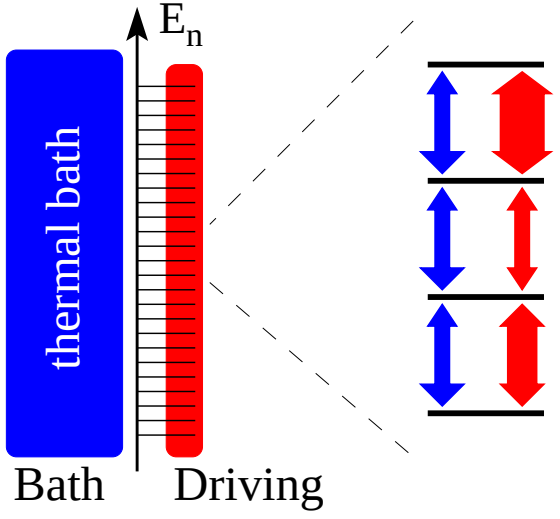


$$w_{nm}^{\text{total}} = w_{nm}^{\beta} + w_{nm}$$

$\nu \equiv$ Driving intensity

Semi-linear response regime ($w_{nm} \gg w^{\beta}$):

$$D[\lambda \boldsymbol{w}] = \lambda D[\boldsymbol{w}]$$
$$D[\boldsymbol{w}^a + \boldsymbol{w}^b] > D[\boldsymbol{w}^a] + D[\boldsymbol{w}^b]$$



For near-neighbor transitions only:

$$D_{\text{semi-linear}} = \left[\overline{1/w_n}\right]^{-1}$$

Linear response regime ($w_{nm} \ll w^{\beta}$):

$$D_{\text{linear}} = \overline{w_n}$$

Experimental implication:

$D\left[\tilde{S}(\omega)\right]$ is linear / semi-linear functional

Response characteristics

$$w_{nm}^{\rm total} \;=\; w_{nm}^{\beta} + w_{nm} \;=\; w_{nm}^{\beta} + \nu g_{nm}$$

$$\frac{w_{nm}^{\beta}}{w_{mn}^{\beta}} \;=\; \exp\left[-\frac{E_n - E_m}{T_B}\right], \qquad g_{nm} \;=\; g_{mn}$$

w_{nm} by themselves - induces diffusion / ergodization

w_{nm}^{β} by themselves - leads to equilibrium

Combined - leads to **NESS**

Linear response and traditional FD: $\nu \times \{g\} \;\ll\; \{w^{\beta}\}$

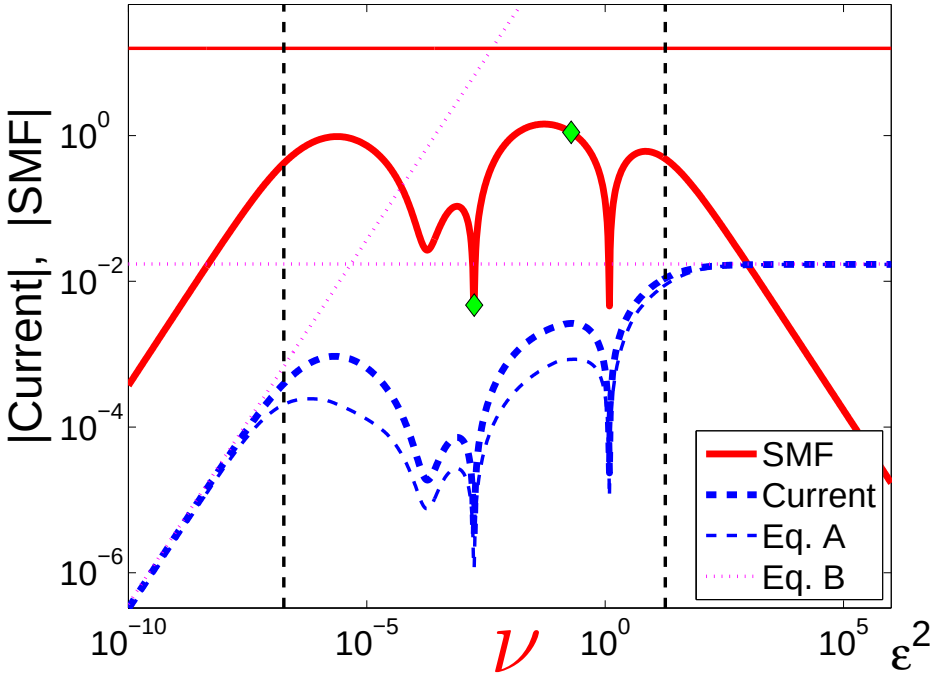
Glassy response and Sinai physics: [within a wide crossover regime]

Semi-linear response and Saturation: $\nu \times \{g\} \;\gg\; \{w^{\beta}\}$

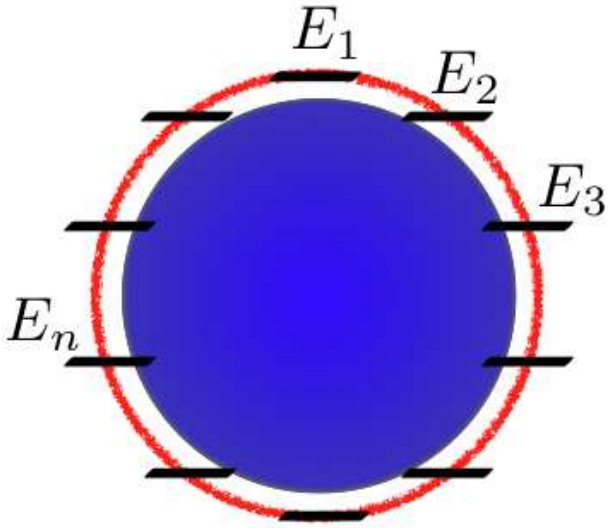
Example: current versus driving

Driving $\rightsquigarrow$ Stochastic Motive Force $\rightsquigarrow$ Current

Regimes: LRT regime, Sinai regime, Saturation regime



$$w^{\text{total}}_{n+1,n} = w^{\beta} + g_n \nu$$



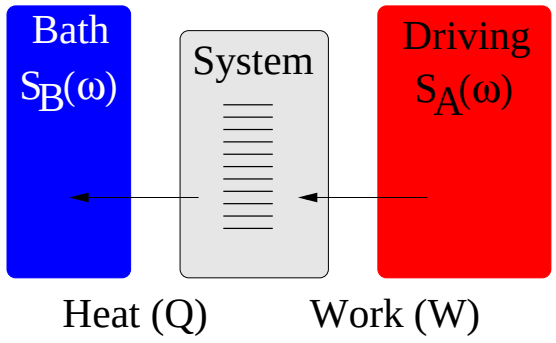
$$I \sim \frac{1}{N}\,\overline{w}\,\exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right]\,2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

The generalized Fluctuation-Dissipation phenomenology

$$w_{nm}^{\text{total}} = w_{nm}^{\beta} + w_{nm}$$

$$\dot{W} = \text{rate of heating} = \frac{D(\nu)}{T_{\text{system}}}$$

$$\dot{Q} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$



Hence at the NESS:

$$T_{\text{system}} = \left(1 + \frac{D(\nu)}{D_B}\right) T_B$$

$$\dot{Q} = \dot{W} = \frac{1/T_B}{D_B^{-1} + D(\nu)^{-1}}$$

Experimental way to extract response:

$$D(\nu) = \frac{\dot{Q}(\nu)}{\dot{Q}(\infty) - \dot{Q}(\nu)} D_B$$

$D(\nu)$ exhibits LRT to SLRT crossover

$$D(\nu) = \left[\overline{\left(\frac{w_n}{w_\beta + w_n}\right)}\right] \left[\overline{\left(\frac{1}{w_\beta + w_n}\right)}\right]^{-1}$$

$$D_{[\text{LRT}]} = \overline{g_n}\,\nu \qquad \text{[weak driving]}$$

$$D_{[\text{SLRT}]} = [\overline{1/g_n}]^{-1}\nu \qquad \text{[strong driving]}$$

Expressions above assume n.n. transitions only.

Digression - derivation of the cooling rate formula

$$\dot{Q} \;=\; \text{cooling rate} \;=\; -\sum_{n,m}(E_n - E_m)\; w^{\beta}_{nm}\; p_m$$

$$p_n - p_m \;=\; \text{occupation imbalance} \;=\; \left[2\tanh\left(-\frac{E_n - E_m}{2T_{nm}}\right)\right]\bar{p}_{nm}$$

$$w^{\beta}_{nm} - w^{\beta}_{mn} \;=\; \text{up/down transitions imbalance} \;=\; \left[2\tanh\left(-\frac{E_n - E_m}{2T_B}\right)\right]\bar{w}^{\beta}_{nm}$$

$$\dot{Q} \;=\; \frac{1}{2}\sum_{n,m}(E_n - E_m)^2\;\frac{\bar{w}^{\beta}_{nm}}{T_B}\bar{p}_{nm} - \frac{1}{2}\sum_{n,m}(E_n - E_m)^2\;\frac{\bar{w}^{\beta}_{nm}}{T_{nm}}\bar{p}_{nm} \;=\; \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

definition of the diffusion coefficient: $$D_B \;\equiv\; \overline{\left[\frac{1}{2}\sum_{n}(E_n - E_m)^2\;\bar{w}^{\beta}_{nm}\right]}$$

definition of effective system temperature: $$\frac{1}{T_{\text{system}}} \;\equiv\; \overline{\left[\frac{1}{T_{nm}}\right]}$$

Conclusions

1. "hopping random-site mode" $\Longrightarrow$ "sparse" matrices
2. "weak quantum chaos" $\Longrightarrow$ "sparse" matrices
3. Applications: beyond the "Drude formula" and beyond the "Wall formula".

4. Resistors network calculation to get the response coefficient.
5. Effective Range Hopping approximation scheme.

6. Semi-linear response if the driving is stronger then the background relaxation.
7. "Glassy NESS" featuring wide distribution of microscopic temperatures.
8. Definition of effective NESS temperature, and extension of the F-D phenomenology.
9. For very strong driving - quantum saturation of the NESS temperature ($T \to T_\infty$).

10. Topological aspects: The emergence of the Sinai regime.
11. Topological term in the formula for the heating rate.

Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding g_c

Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding g_s

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

Random networks: Mott; Miller, Abrahams; Ambegaokar, Halperin, Langer; Pollak [...]

Random site model: Alexander, Bernasconi, Schneider, Orbach; Amir, Oreg, Imry [...]

Extensions related to: Sinai; Derrida,Pomeau; Burlatsky, Oshanin, Mogutov, Moreau, [...]

[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)
[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)
[3] S. Bandopadhyay, Y. Etzioni, D. Cohen (EPL 2006)
[4] M. Wilkinson, B. Mehlig, D. Cohen (EPL 2006)
[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos, D. Cohen (JPA/FTC 2008)
[6] A. Stotland, T. Kottos, D. Cohen (PRB 2010)
[7] A. Stotland, D. Cohen, N. Davidson (EPL 2009)
[8] A. Stotland, L.M. Pecora, D. Cohen (EPL 2010, PRE 2011)
[9] D. Hurowitz, D. Cohen (EPL 2011)
[10] Yaron de Leeuw, D. Cohen (PRE 2012)