Celebrating 70th birthday of Rick Heller

Strong localization and the exploration of phase space for quantum thermalization

Doron Cohen, Ben-Gurion University

[1] Christine Khripkov, Amichay Vardi, DC [**preprint, 2017**]
[2] Christine Khripkov, Amichay Vardi, DC [**New Journal of Physics 2015**]
[3] DC, Slava Yukalov, Klaus Ziegler [**Physical Review A (2016)**]
[4] Igor Tikhonenkov, Amichay Vardi, James Anglin, DC [**Physical Review Letters 2013**]
[5] Geva Arwas, DC [**Physical Review B 2017**]
[6] Geva Arwas, DC [**New Journal of Physics 2016**]
[7] Geva Arwas, Amichay Vardi, DC [**Scientific Reports 2015**]

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Quantum localization and the rate of exploration of phase space

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(Received 29 August 1986)

The "quantum exploration" notion of Heller

The LDOS: $\rho(E) = \sum p_\alpha \delta(E - E_\alpha)$

Δ_0 = The mean level spacing

Δ_E = The width of the energy shell

$\mathcal{N}_E$ = States within the energy shell

$\mathcal{N}_\infty$ = Participating states

$\mathcal{F}^{\rm qm}$ = Localization measure

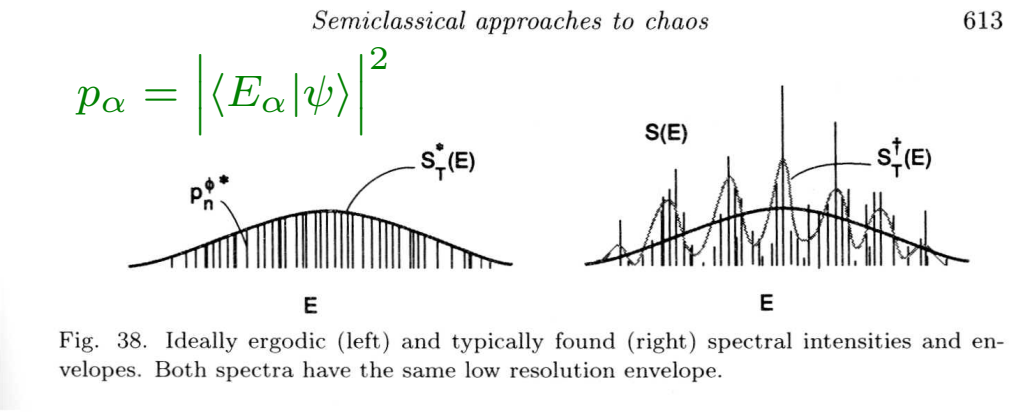


Fig. 38. Ideally ergodic (left) and typically found (right) spectral intensities and envelopes. Both spectra have the same low resolution envelope.

$$t_H = \frac{2\pi}{\Delta_0}, \qquad t_E = \frac{2\pi}{\Delta_E}, \qquad \mathcal{N}_E = \frac{\Delta_E}{\Delta_0}, \qquad \mathcal{N}_\infty = \left[\sum_\alpha p_\alpha^2\right]^{-1}, \qquad \mathcal{F}^{\rm qm} \equiv \frac{\mathcal{N}_\infty}{\mathcal{N}_E}$$

The number of states that participate in the dynamics up to time t is:

$$\mathcal{N}_t \equiv \left\{ \text{trace}\left[\overline{\rho}(t)^2\right] \right\}^{-1} = \left[\frac{2}{t}\int_0^t \left(1 - \frac{\tau}{t}\right)\mathcal{P}(\tau)d\tau\right]^{-1} \qquad \overline{\rho}(t) \equiv \frac{1}{t}\int_0^t \rho(t')dt'$$

Short times: $\mathcal{N}_t^{\rm qm} \approx \mathcal{N}_t^{\rm sc} \approx t/t_E$ (based on the classical envelope)

Long times: $\mathcal{N}_t^{\rm cl} \to \mathcal{N}_\infty$ (due to the discreteness of the spectrum)

The "classical exploration" notion of random walk

Spreading (semiclassical or quantum):

$$\Omega_t^{\mathrm{sc/qm}} \quad = \quad \left\{ \sum_r \left[P_t(r|r_0) \right]^2 \right\}^{-1} \quad \equiv \quad \mathrm{PN}\left\{ [r_0], t \right\}$$

Classical exploration:

$$\Omega_t^{\mathrm{cl}} \quad = \quad \mathrm{PN}\left\{ r_0, [0, t] \right\}$$

Which can be written as

$$\Omega_t^{\mathrm{cl}} \quad \equiv \quad \left\{ \mathrm{trace}\left[\overline{\rho}_{\mathrm{cl}}(t)^2 \right] \right\}^{-1}$$

Example:

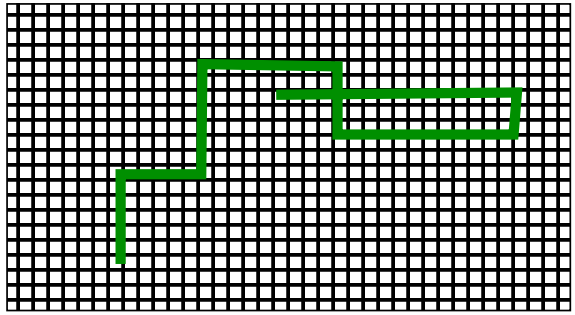
Random walk in 3D

$t = 100$ steps

explored volume ~ 99

spreading radius ~ 10

spreading volume ~ 1000



Classical exploration for random walk on a lattice [Montroll and Weiss 1965]:

$$\Omega_t \sim \sqrt{D_0 t} \qquad \text{for } d = 1$$

$$\Omega_t \sim \frac{v_0 t}{\log(t)} \qquad \text{for } d = 2$$

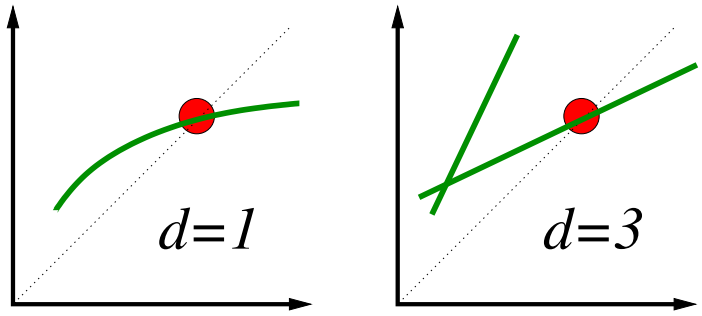
$$\Omega_t \sim v_0 t \qquad \text{for } d > 2$$

The breaktime concept

- Stationary view of strong localization: interference of trajectories.
- Scaling theory of localization: the importance of dimensionality.
- Dynamical view of strong localization: breakdown of quantum-classical correspondence.

$$t_H[\Omega] \;=\; \frac{2\pi}{\Delta_0} \;\propto\; \Omega$$

$$t \;\ll\; t_H[\Omega_t] \qquad \rightsquigarrow \qquad t^*$$



$\Omega_t \;=\; \sqrt{D_0 t}$ for $d = 1$ $\rightsquigarrow$ always localization

$\Omega_t \;=\; c_0 + v_0 t$ for $d > 2$ $\rightsquigarrow$ mobility edge

For diffusion in 1D we get $\xi = gD$, where g is the local DOS.

Chirikov, Izrailev, Shepelyansky [SovSciRevC 1981]; Shepelyansky [PhysicaD 1987];

Dittrich, *Spectral statistics for 1D disordered systems* [Phys Rep 1996];

DC, *Periodic Orbits Breaktime and Localization* [JPA 1998].

Manifestation of localization in thermalization?

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \varepsilon}\left(g(\varepsilon)D(\varepsilon)\frac{\partial}{\partial \varepsilon}\left(\frac{1}{g(\varepsilon)}\rho\right)\right)$$

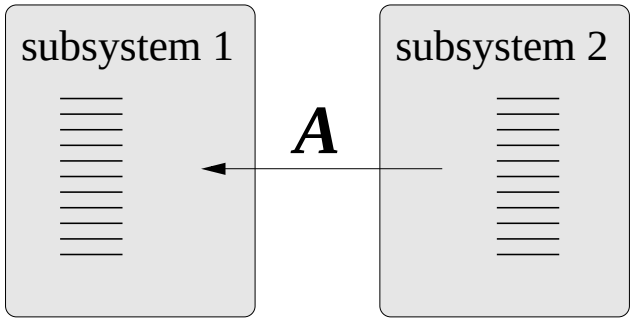
$g(\varepsilon)$ - local density of states

Rate of energy transfer [FPE version]:

$A(\varepsilon) = \partial_\varepsilon D + (\beta_1 - \beta_2)D$

For canonical preparation:

$$\langle A(\varepsilon)\rangle = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\langle D(\varepsilon)\rangle$$



Later we consider a Bose-Hubbard system where the diffusion is in x

$x =$ the occupation of subsystem 1

$N - x =$ the occupation of subsystem 2

Hurowitz, DC (EPL 2011) - MEQ version

Tikhonenkov, Vardi, Anglin, DC (PRL 2013)] - FPE version

Bunin, Kafri (JPA 2013) - NFT version

Khripkov, Vardi, DC (NJP 2015) - Resistor network calculation of $D(\varepsilon)$

Question: Do we have $\xi = g(\varepsilon)D(\varepsilon)$?

Phase space formulation of the QCC condition

We propose a generalized QCC condition for the purpose of breaktime determination:

Rough version:
$$t \;<\; \left[\frac{\Omega_t^{\rm cl}}{\Omega_E}\right] t_H$$

Refined version:
$$\mathcal{N}_t^{\rm sc} \;<\; \mathcal{F}_{\rm erg}^{\rm qm} \left[\frac{\mathcal{N}_E}{\Omega_E}\right] \Omega_t^{\rm cl}$$

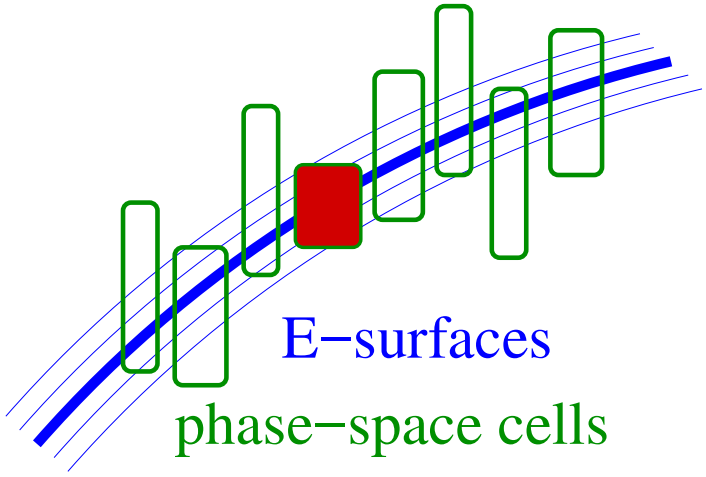
$\mathcal{N}_E$ = total number of states within the energy shell (r_0 dependent)

$\mathcal{F}_{\rm erg}^{\rm qm}$ = filling fraction for a quantum ergodic state, say $= 1/3$

Ω_E = number of cells that intersect an energy-surface

$\Omega_t^{\rm cl}$ = explored phase-space volume during time t (starting at r_0)

$\mathcal{N}_t^{\rm sc} \approx t/t_E$ = semiclassical number of participating-states during time t



$$\left|\langle r_j | E_\alpha \rangle\right|^2$$

It is unavoidable to use in the semiclassical analysis improper Planck cells. Namely, a chaotic eigenstate is represented by a microcanonical energy-shell of thickness $\propto \hbar^d$ and radius $\propto \hbar^0$. For some preparations it is implied that $\mathcal{N}_E \ll \Omega_E$ rather than $\mathcal{N}_E \sim \Omega_E$.

Cartoon: $\Omega_E = 8$, while $\mathcal{N}_E = 5$.

Proper Planck cell: $\Delta Q \Delta P > \hbar/2$ for each coordinate.

The Bose Hubbard Hamiltonian

The system consists of N bosons in M sites. Later we add a gauge-field Φ.

$$\mathcal{H}_{\rm BHH} = \frac{U}{2}\sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)$$

$$u \equiv \frac{NU}{K} \qquad \text{[classical, stability, supefluidity, self-trapping]}$$

$$\gamma \equiv \frac{Mu}{N^2} \qquad \text{[quantum, Mott-regime]}$$

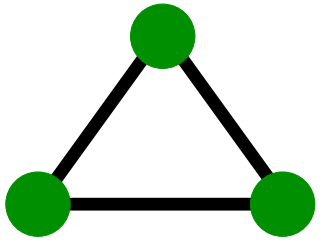
Dimer (M=2): Minimal BHH; Bosonic Josephson junction; Pendulum physics [1,5].
Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].
Trimer (M=3): Minimal model for low-dimensional chaos; Coupled pendula physics.
Triangular trimer (M=3): Minimal model with topology, Superfluidity [6], Stirring [7].
Larger rings (M>3) High-dimensional chaos; web of non-linear resonances [7].
Coupled subsystems (M>3): Minimal model for Thermalization [8,9].

[1] Chuchem, Smith-Mannschott, Hiller, Kottos, Vardi, DC (PRA 2010).
[2] Smith-Mannschott, Chuchem, Hiller, Kottos, DC (PRL 2009).
[3] Boukobza, Moore, DC, Vardi (PRL 2010).
[4] Khripkov, Vardi, DC (PRA 2012)
[5] Khripkov, DC, Vardi (JPA 2013, PRE 2013).
[6] Geva Arwas, Vardi, DC (PRA 2014, SREP 2015, NJP 2016, PRB 2017).
[7] Hiller, Kottos, DC (EPL 2008, PRA 2008).
[8] Tikhonenkov, Vardi, Anglin, DC, (PRL 2013).
[9] Christine Khripkov, Vardi, DC (NJP 2015).

Quantum Chaos perspective on Metastability and Ergodicity

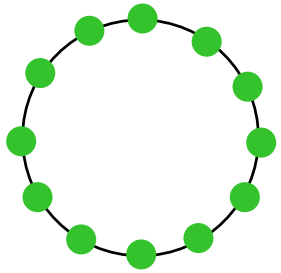
Stability of flow-states (I):

- Landau stability of flow-states ("Landau criterion")
- Bogoliubov perspective of dynamical stability
- KAM perspective of dynamical stability



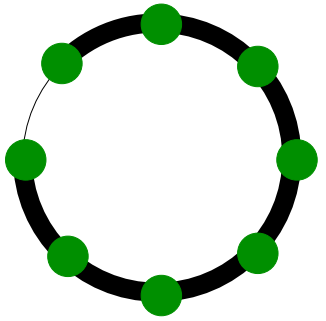
Stability of flow-states (II):

- Considering high dimensional chaos ($M > 3$).
- Web of non-linear resonances.
- Irrelevance of the the familiar Beliaev and Landau damping terms.
- Analysis of the quench scenario.



Coherent Rabi oscillations:

- The hallmark of coherence is Rabi oscillation between flow-states.
- Ohmic-bath perspective $\rightsquigarrow$ $\eta = (\pi/\sqrt{\gamma})$
- Feasibility of Rabi oscillation for $M < 6$ devices.
- Feasibility of of chaos-assisted Rabi oscillation.



Thermalization:

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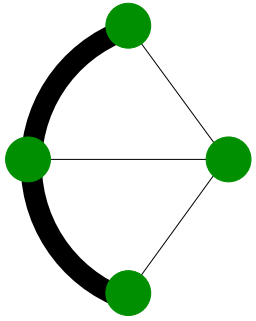
- Resistor-Network calculation of the diffusion coefficient.
- Observing regions with Semiclassical Localization.
- Observing regions with Dynamical Localization.

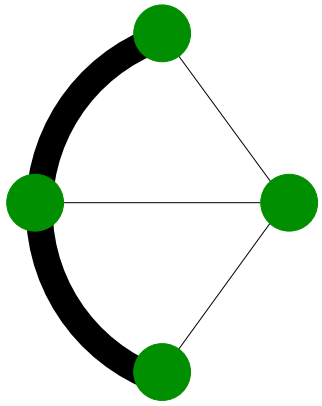


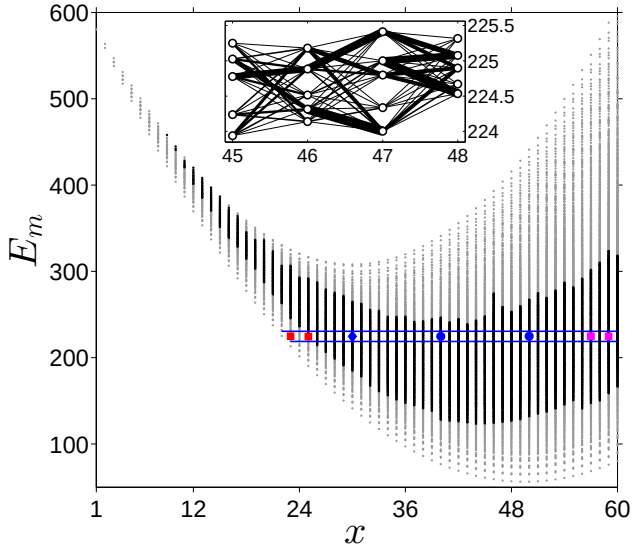
The minimal model for thermalization

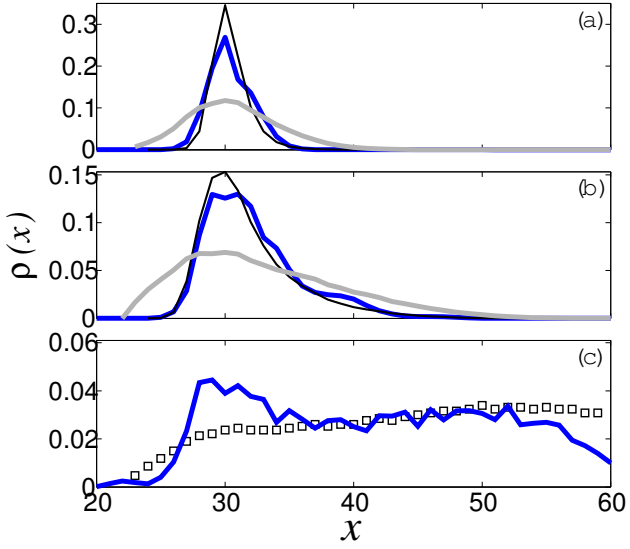
The FPE description makes sense if the sub-systems are chaotic.

Minimal model for a chaotic sub-system: BHH trimer.

Minimal model for thermalization: BHH trimer + monomer







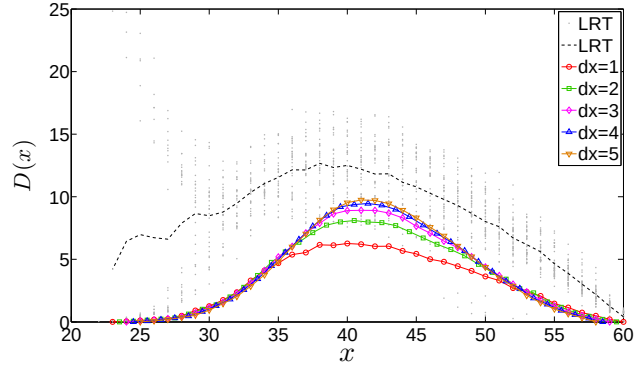
$N = 60 =$ number of particles

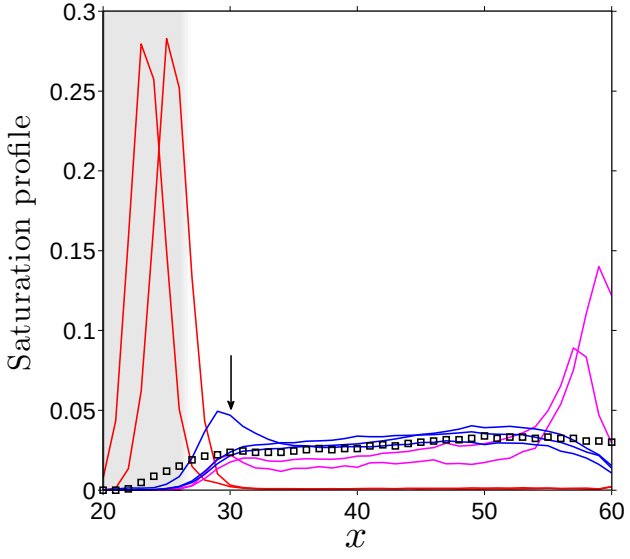
$x =$ occupation of the trimer

$N - x =$ occupation of the monomer

$\rho(x) =$ probability distribution

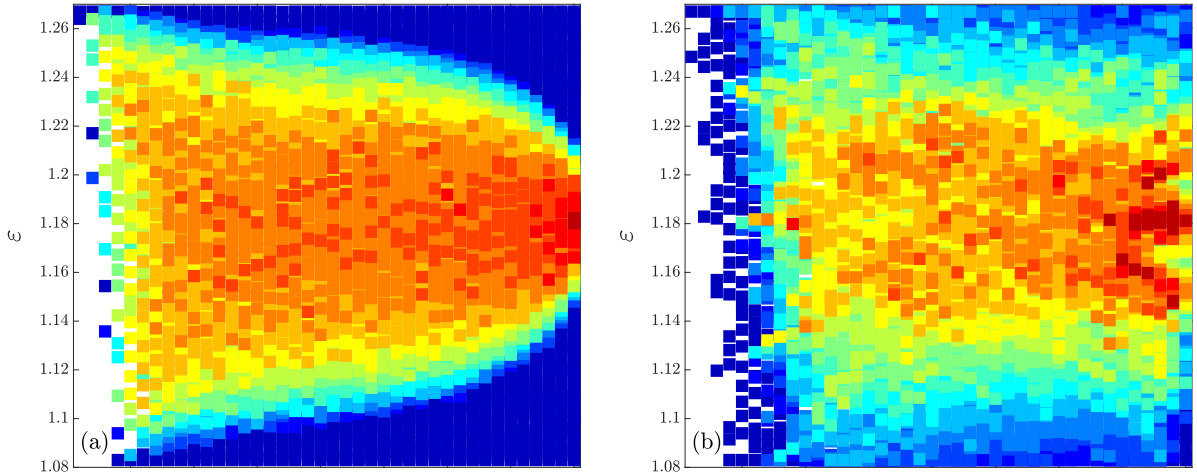
$$\frac{\partial \rho(x)}{\partial t} = \frac{\partial}{\partial x}\left[g(x) D(x) \frac{\partial}{\partial x}\left(\frac{\rho(x)}{g(x)} \right) \right]$$





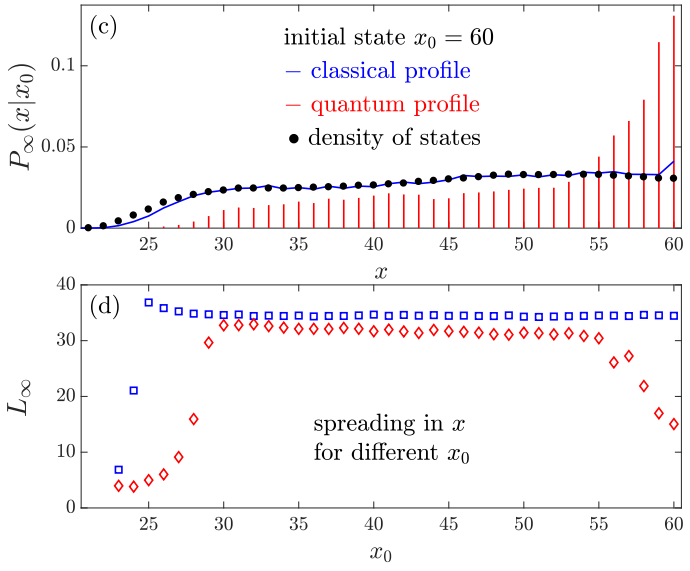
Dynamical localization

Here we start the simulation with $x_0 = 60$,
meaning that initially all the particles are in the trimer.
We plot the saturation profile $P_\infty(x, \varepsilon)$.



Classical

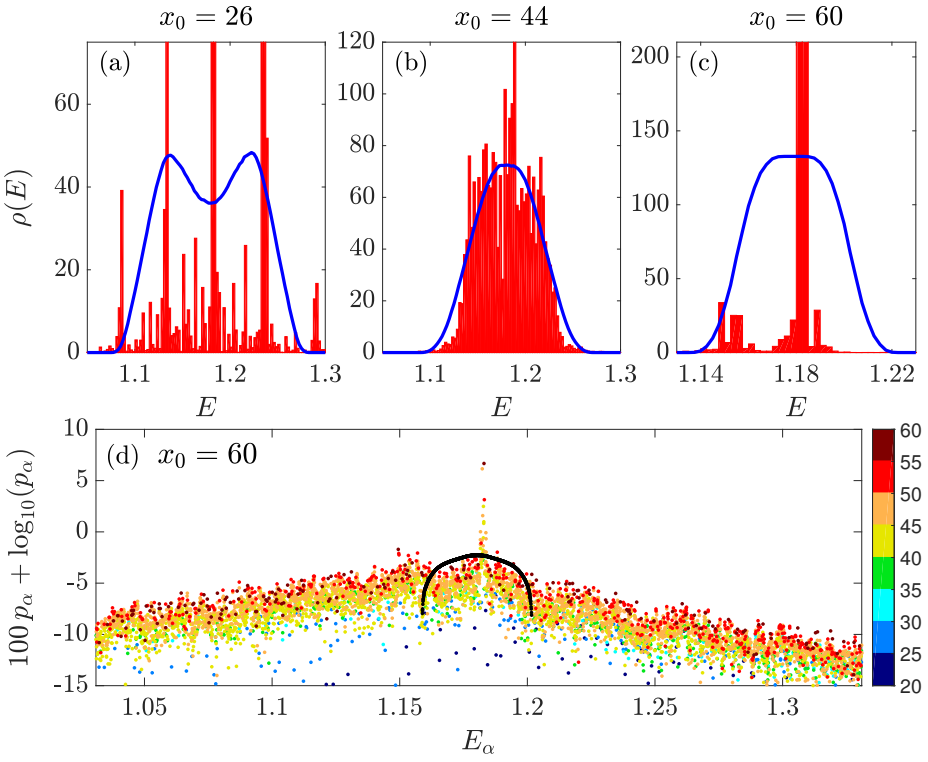
Quantum

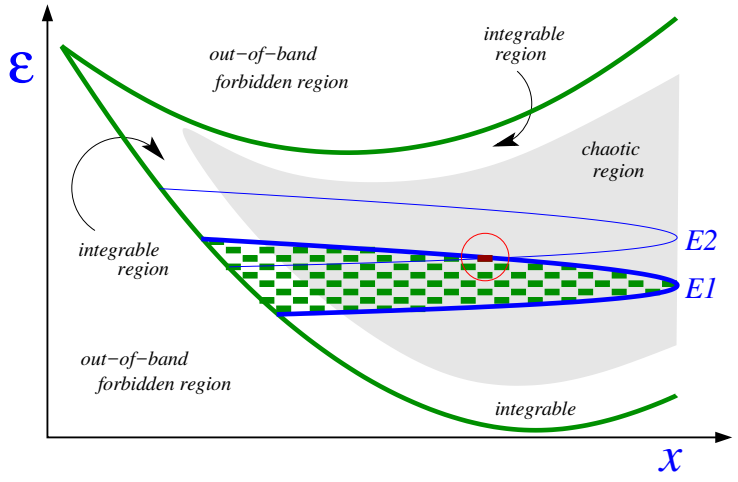


The LDOS of the initial preparation

We observed dynamical localization if we start with $x_0 < 30$ or with $x_0 > 55$.

Let us look on the LDOS of representative preparations:

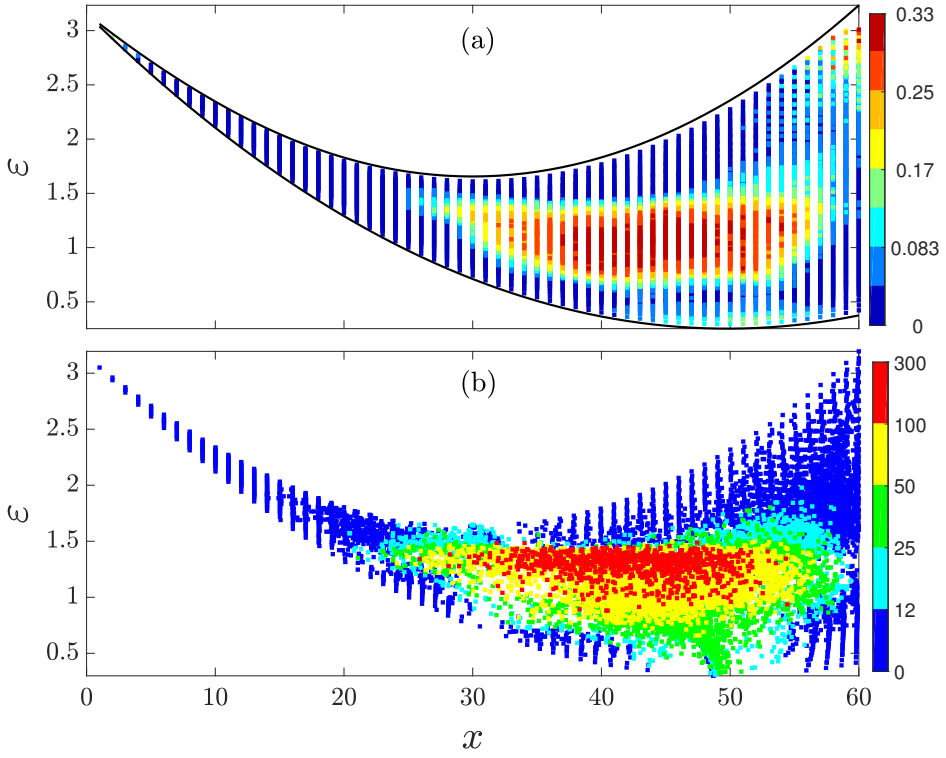




Localization does not always manifests as sparsity. It depends on the geometry of the $r = (x, \epsilon)$ space.

$$\mathcal{F}^{\rm qm} \equiv \frac{\mathcal{N}_\infty}{\mathcal{N}_E}$$

Localization of the eigenstates



Upper panel:

The unperturbed states $|r\rangle = |x, \varepsilon\rangle$,
color-coded according to $\mathcal{F}^{\mathrm{qm}}$.
arranged by (x, ε).

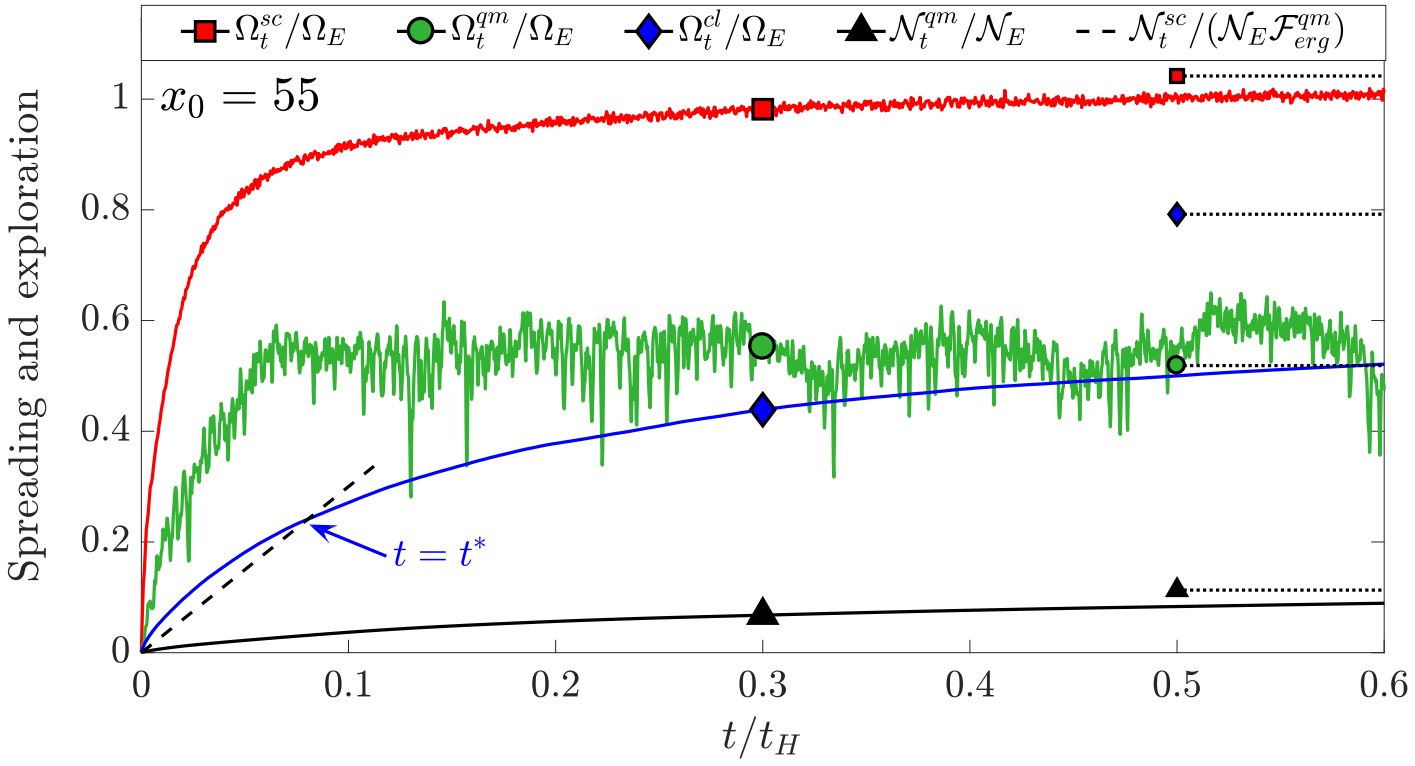
Lower panel:

The perturbed states $|E_\alpha\rangle$,
color-coded according to $\mathrm{var}(x)_\alpha$,
sorted by $(\langle x\rangle_\alpha, \langle \varepsilon\rangle_\alpha)$.

Participation, Exploration, and Breaktime

We display $\Omega^{\rm cl}(t)$, and $\Omega^{\rm sc}(t)$, and $\Omega^{\rm qm}(t)$, and $\mathcal{N}(t)$.

The breaktime is determined by the intersection of the scaled $\mathcal{N}^{\rm sc}(t)$ with the scaled $\Omega^{\rm cl}(t)$.

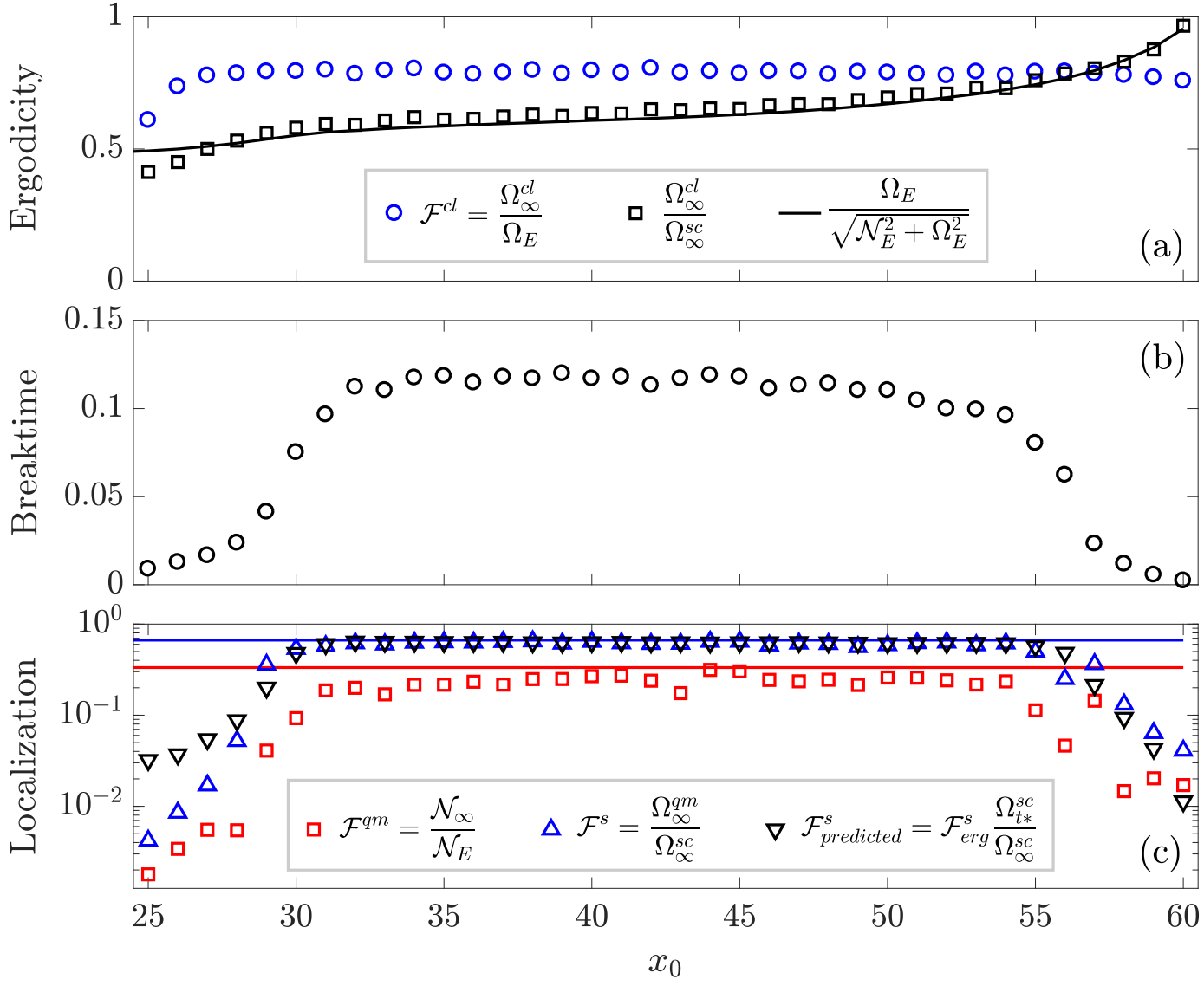


$$\mathcal{F}^{\rm s} \equiv \frac{\Omega_\infty^{\rm qm}}{\Omega_\infty^{\rm sc}}$$

Prediction:

$$\Omega_\infty^{\rm qm} \approx \mathcal{F}_{\rm erg}^{\rm qm}\, \Omega_{t^*}^{\rm sc}$$

Localization measures



$$\mathcal{F}^{\mathrm{cl}} \equiv \frac{\Omega^{\mathrm{cl}}_\infty}{\Omega_E}$$

$$\mathcal{F}^{\mathrm{qm}} \equiv \frac{\mathcal{N}^{\mathrm{qm}}_\infty}{\mathcal{N}_E}$$

$$\mathcal{F}^{\mathrm{s}} \equiv \frac{\Omega^{\mathrm{qm}}_\infty}{\Omega^{\mathrm{sc}}_\infty}$$

For ergodic system

$$\Omega^{\mathrm{cl}}_\infty \rightsquigarrow \Omega_E$$

$$\Omega^{\mathrm{sc}}_\infty \rightsquigarrow \sqrt{[\mathcal{N}_E(r_0)]^2 + \Omega_E^2}$$

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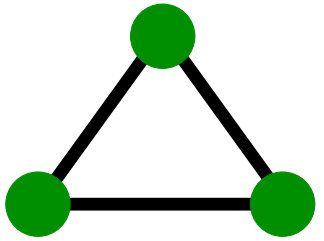

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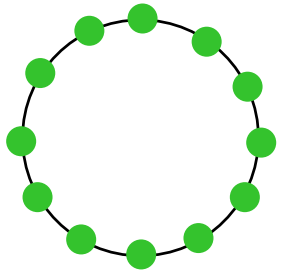

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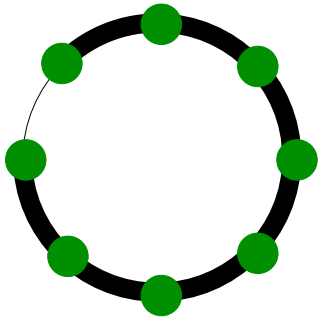

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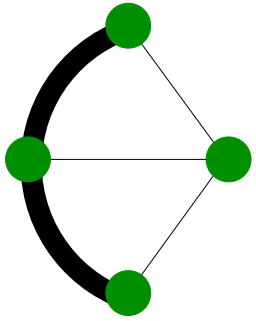
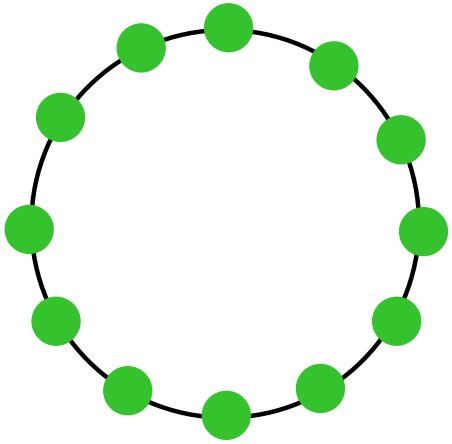


The Model (non-rotating ring)

A Bose-Hubbard system with M sites and N bosons:

$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)\right]$$



In a semi-classical framework:

$$a_j \;=\; \sqrt{\boldsymbol{n}_j}\;\mathrm{e}^{i\boldsymbol{\varphi}_j} \quad , \quad [\boldsymbol{\varphi}_j, \boldsymbol{n}_i] \;=\; i\delta_{ij}$$

$$z \;=\; (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_M, \;\; \boldsymbol{n}_1, \cdots, \boldsymbol{n}_M)$$

This is like M coupled oscillators with $\mathcal{H} = H(z)$

$$H(z) \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}\boldsymbol{n}_j^2 - K\sqrt{\boldsymbol{n}_{j+1}\boldsymbol{n}_j}\cos\left(\boldsymbol{\varphi}_{j+1} - \boldsymbol{\varphi}_j\right)\right]$$

The dynamics is generated by the Hamilton equation:

$$\dot{z} \;=\; \mathbb{J}\boldsymbol{\partial} H \quad , \quad \mathbb{J} \;=\; \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

(DNLS)

Classically there is a single dimensionless parameter:

$$u \;=\; \frac{NU}{K}$$

Rescaling coordinates:

$$\tilde{\boldsymbol{n}} \;=\; \boldsymbol{n}/N$$

$$[\boldsymbol{\varphi}_j, \tilde{\boldsymbol{n}}_i] \;=\; i\frac{1}{N}\delta_{ij}$$

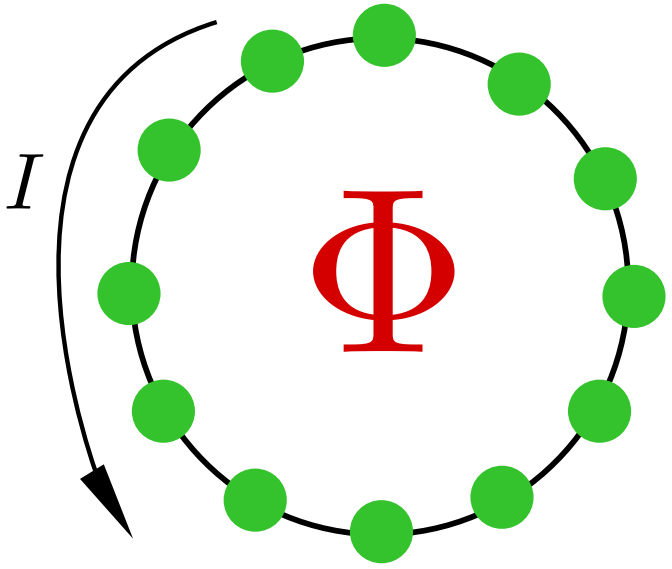
$$\gamma \;\equiv\; \frac{\mathsf{m}^* g}{\rho} \;=\; \frac{Mu}{N^2}$$

The model (rotating ring)

In the rotating reference frame we have a Coriolis force,
which is like magntic field $\mathcal{B} = 2\mathsf{m}\Omega$. Hence is is like having flux

$$\Phi \;=\; 2\pi R^2 \mathsf{m}\;\Omega$$

Note: there are optional experimental realizations.



$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\left(\mathrm{e}^{i(\Phi/M)}a_{j+1}^\dagger a_j + \mathrm{e}^{-i(\Phi/M)}a_j^\dagger a_{j+1}\right)\right]$$

Summary of model parameters:

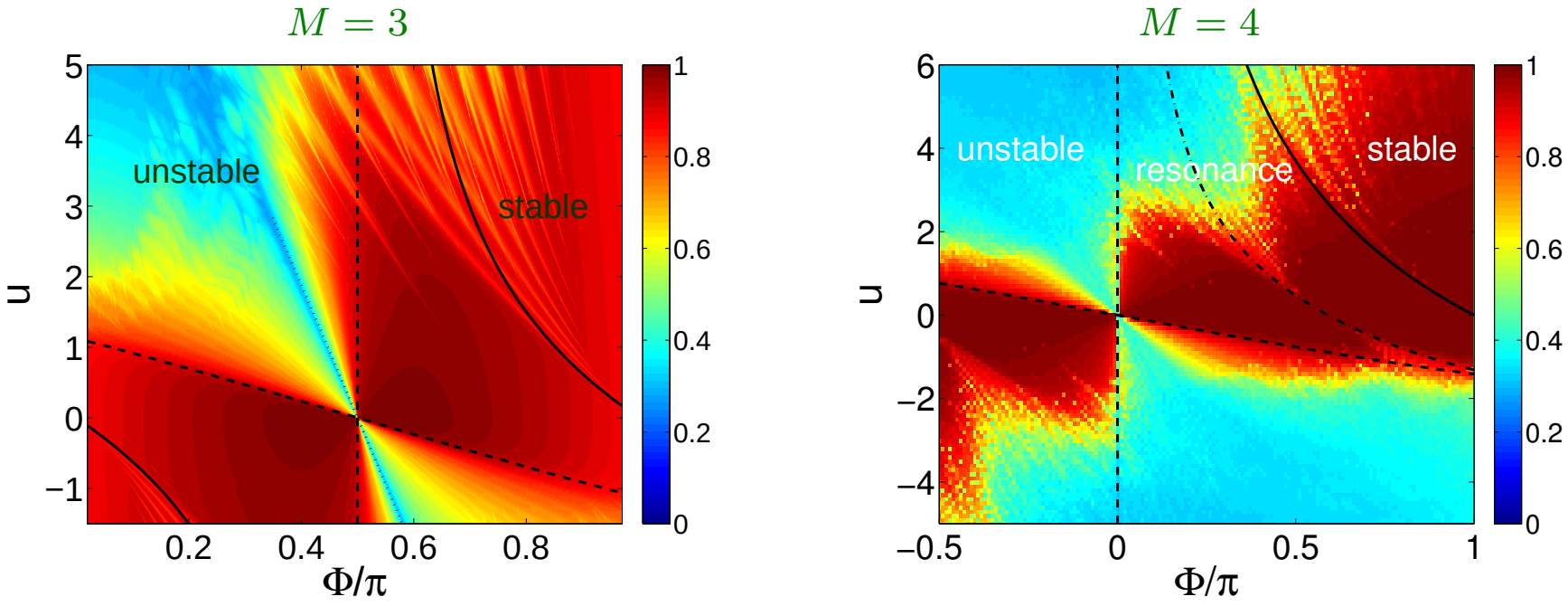
The "classical" dimensionless parameters of the DNLS are u and Φ.
The number of particles N is the "quantum" parameter (optionally γ).
The system has effectively $d = M-1$ degrees of freedom.

$M = 2$	Bosonic Josephson junction (Integrable)
$M = 3$	Minimal circuit (mixed chaotic phase-space)
$M > 3$	High dimensional chaos (Arnold diffusion)
$M \to \infty$	Continuous ring (Integrable)

Flow-state stability regime diagram

The I of the maximum current state is imaged as a function of (Φ, u)

- solid lines = energetic stability borders (Landau)
- dashed lines = dynamical stability borders (Bogoliubov)



The traditional paradigm associates flow-states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov linear-stability-analysis, are used to determine the viability of superfluidity.

Energetic vs Dynamical stability

Poincare section $n_2 = n_3$ at the flow-state energy.

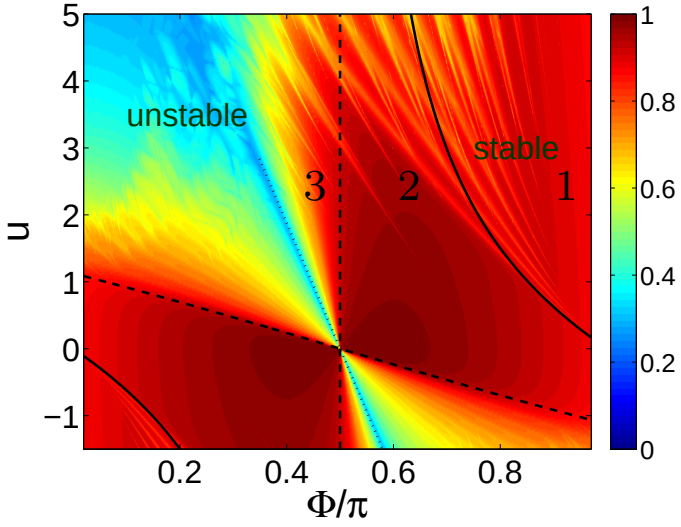
(1) Energetic stability; **(2)** Dynamical stability.

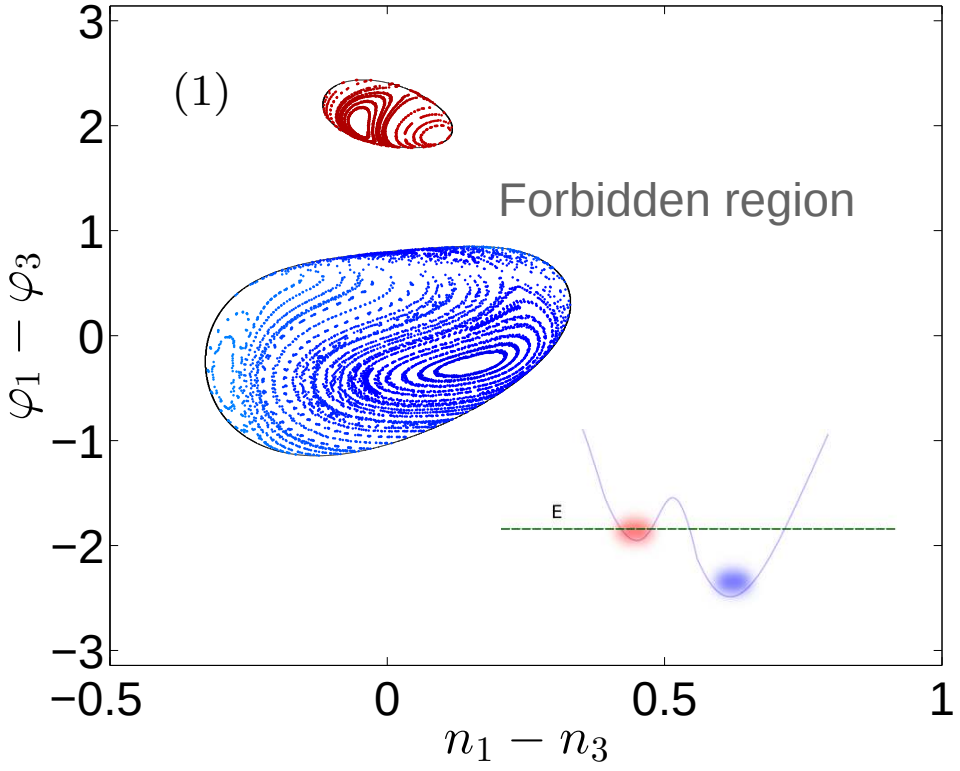
red trajectories = large positive current

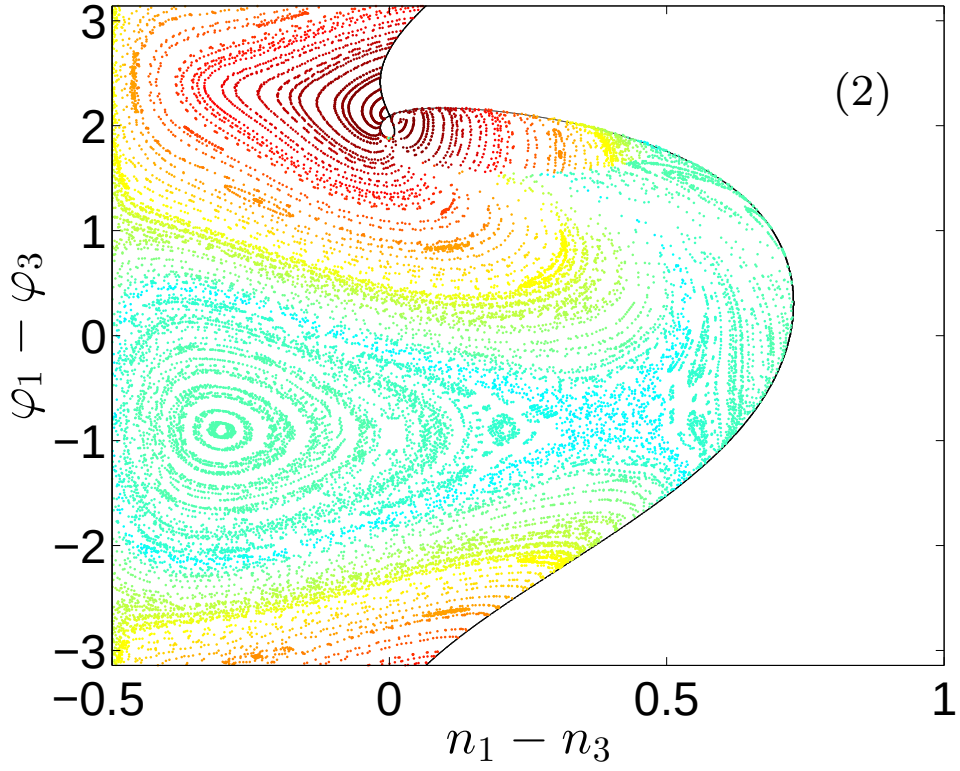
blue trajectories = large negative current

The flow-state fixed-points are located along the symmetry axis:

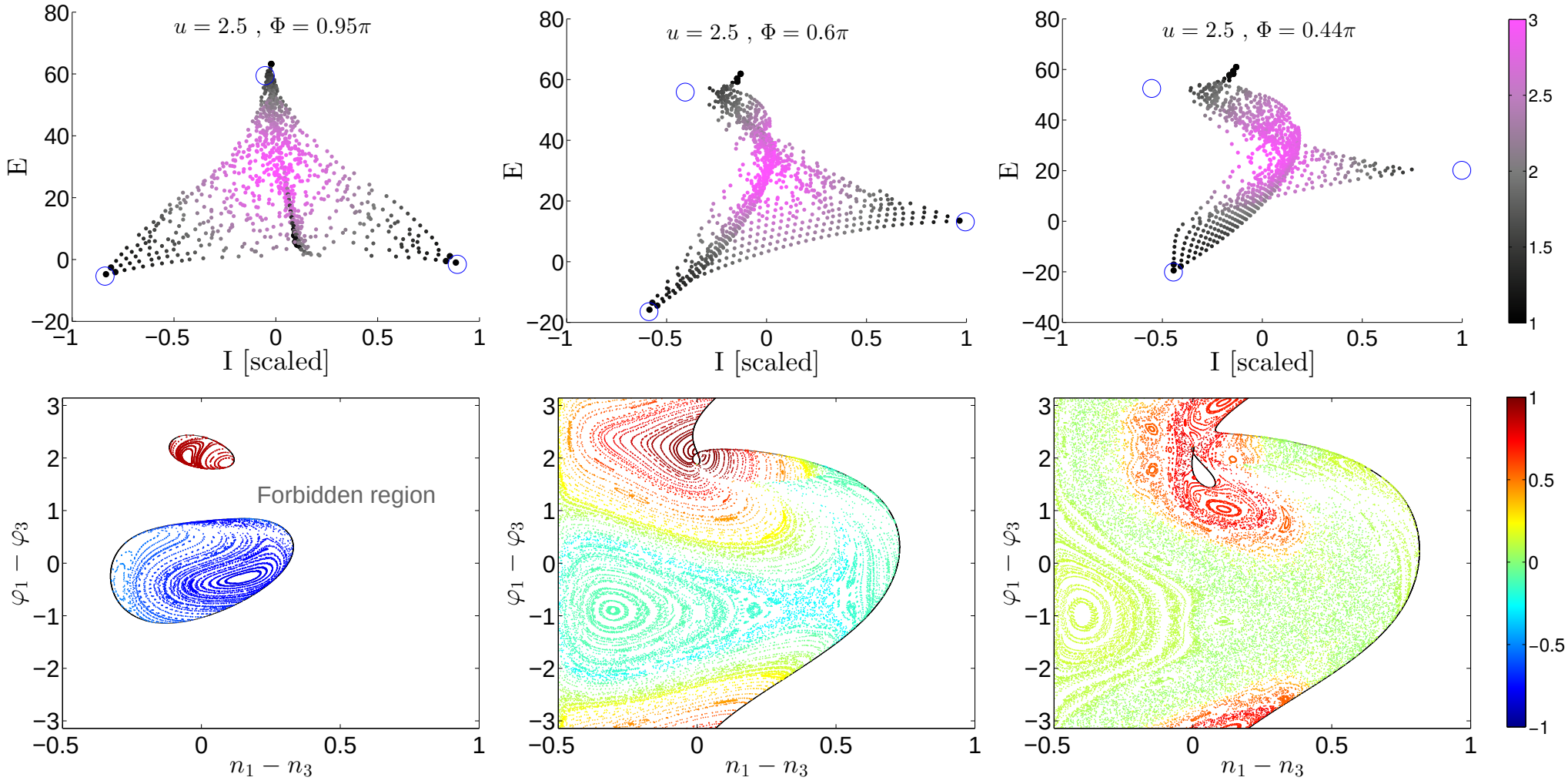
$$n_1 = n_2 = \cdots = N/M, \qquad \varphi_i - \varphi_{i-1} = \left(\frac{2\pi}{M}\right) m$$







KAM stability - elliptic islands and chaotic ponds

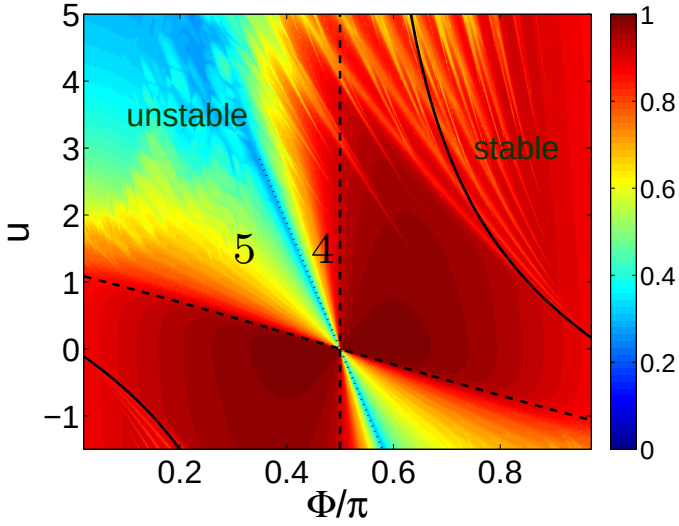


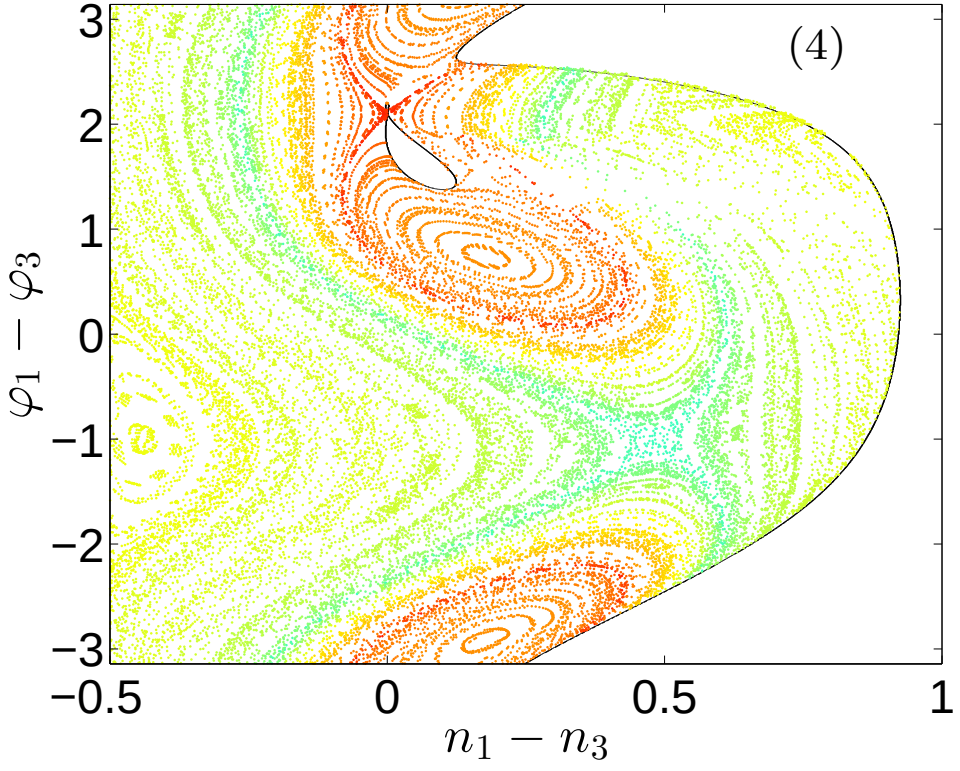
Swap transition

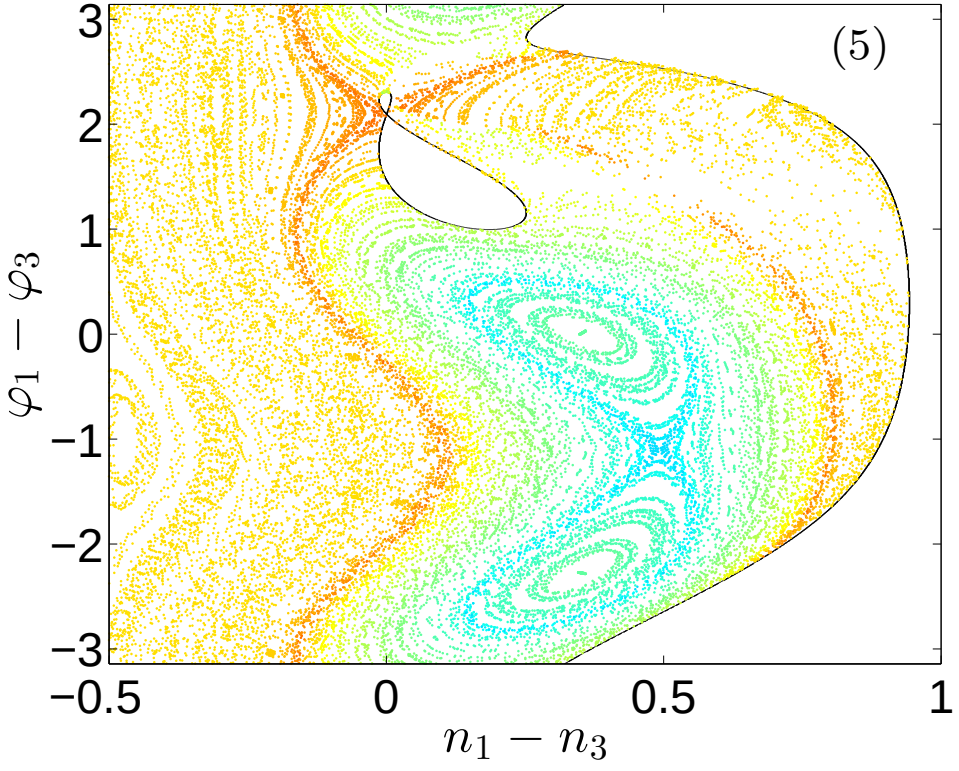
In **(4)** and **(5)** dynamical stability is lost $\rightsquigarrow$ chaotic motion.
But the chaotic trajectory is confined within a chaotic pond;
uni-directional chaotic motion; superfluidity persists!
At the separatrix swap-transition superfluidity diminishes.

Swap transition (dotted line):

$$u \;=\; 18\sin\left(\frac{\pi}{6}-\frac{\Phi}{3}\right)$$







Manifestation of phase space topology for $M > 3$ circuits

Number of freedoms: $d = (M-1)$

$d = 2$ Mixed phase space: islands, ponds, and chaotic sea

$d > 2$ High dimensional chaos: Arnold web and chaotic sea

- The energy surface is $2d - 1$ dimensional
- KAM tori are d dimensional
- The KAM tori are not effective in blocking the transport on the energy shell if $d > 2$.
- Resonances form an "Arnold Web" $\rightsquigarrow$ "Arnold diffusion"
- As u becomes larger this non-linear leakage effect is enhanced, stability of the motion is deteriorated, and the current is diminished.

For $M = 3$ the 3 dimensional energy surface is divided into territories by the 2 dimensional KAM tori.

For $M = 4$ the 5 dimensional energy surface cannot be divided into territories by the 3 dimensional KAM tori.

Non-linear resonances

Regime diagram for flow-state metastability:

- Via quantum eigenstates
- Via quantum quench simulation
- Via semiclassical simulation

Observation:

The linear-stability analysis of Bogoliubov is not a sufficient condition for strict dynamical stability. A non-linear resonance between the frequencies can destroy the dynamical stability.

The "1:2" resonance

for the $m = 1$ flow-state of M=4 ring:

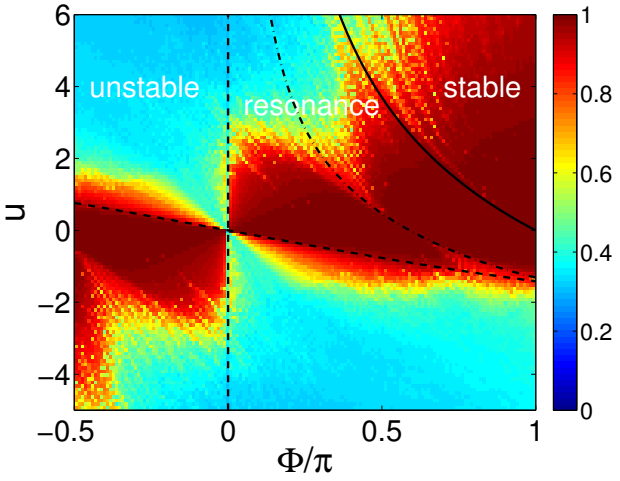
$$u = 4\cot\left(\frac{\Phi}{4}\right)\left[3\cos\left(\frac{\Phi}{4}\right) - \sqrt{6 + 2\cos\left(\frac{\Phi}{2}\right)}\right]$$

Addressing all flow-states in one diagram:

$$\phi = \Phi - 2\pi m = \text{unfolded phase} \in [-M\pi, M\pi]$$

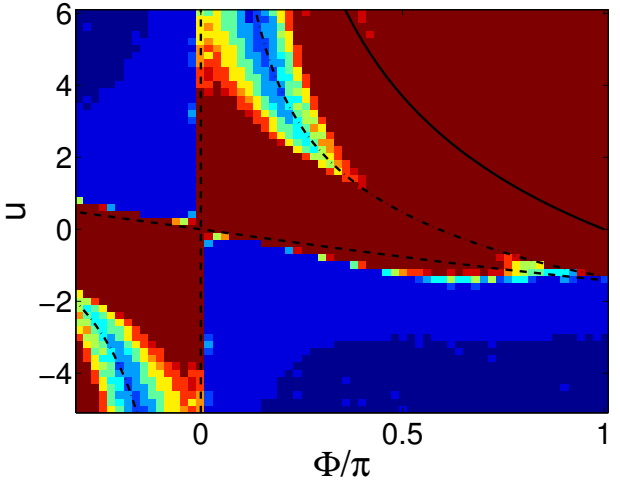
Quantum regimes $M = 4$

$N = 32$ particles



Classical reconstruction

Small cloud



The non-linear terms

$$\mathcal{H} = \sum_k \epsilon_k \boldsymbol{b}_k^\dagger \boldsymbol{b}_k + \frac{U}{2M} \sum_{\langle k_1..k_4 \rangle} \boldsymbol{b}_{k_4}^\dagger \boldsymbol{b}_{k_3}^\dagger \boldsymbol{b}_{k_2} \boldsymbol{b}_{k_1}$$

Assuming condensation at the k=0 orbital
the Hamiltonian can be expressed in terms
of Bogoliubov quasi-particles creation operators:

$$\boldsymbol{b}_q^\dagger = u_q \boldsymbol{c}_q^\dagger + v_q \boldsymbol{c}_{-q}$$

$$q = \frac{2\pi}{M} m$$

$m = \text{integer} \neq 0$

$$\frac{M}{2} < m \leq \frac{M}{2}$$

Approximated Hamiltonian at the vicinity of the condensate:

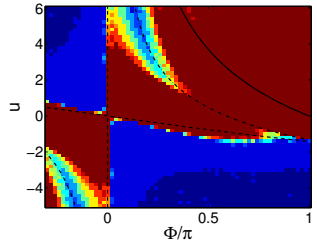
$$\mathcal{H} = \sum_q \omega_q \boldsymbol{c}_q^\dagger \boldsymbol{c}_q + \frac{\sqrt{N}U}{M} \sum_{\langle q_1, q_2 \rangle} \left[A_{q_1,q_2} \left(\boldsymbol{c}_{-q_1-q_2} \boldsymbol{c}_{q_2} \boldsymbol{c}_{q_1} + \text{h.c.} \right) + B_{q_1,q_2} \left(\boldsymbol{c}_{q_1+q_2}^\dagger \boldsymbol{c}_{q_2} \boldsymbol{c}_{q_1} + \text{h.c.} \right) \right]$$

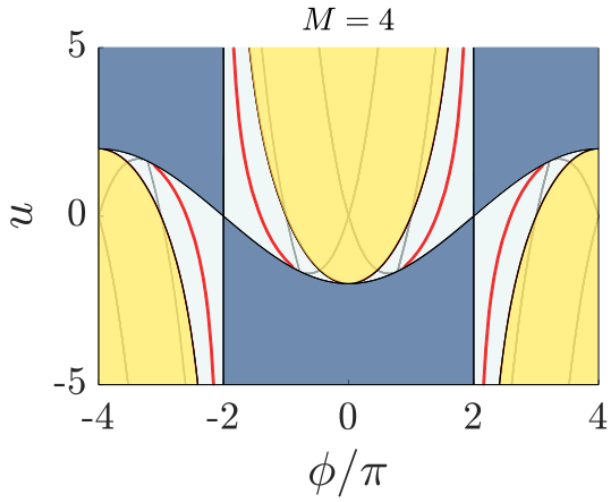
- The "B" terms are the Beliaev and Landau damping terms. [gray lines]
- The "A" terms are usually ignored. [red lines]

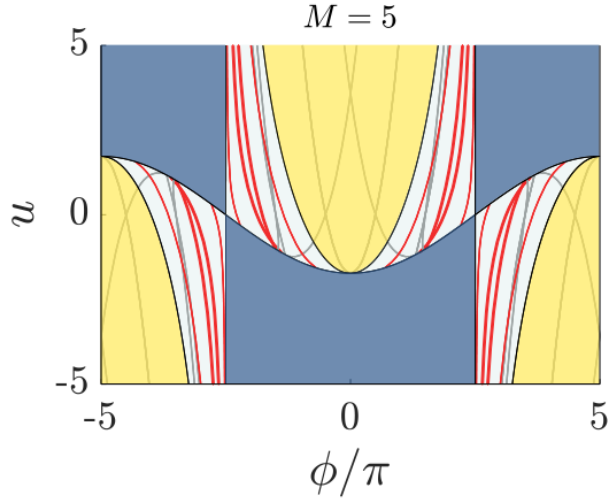
Mapping the non-linear resonances

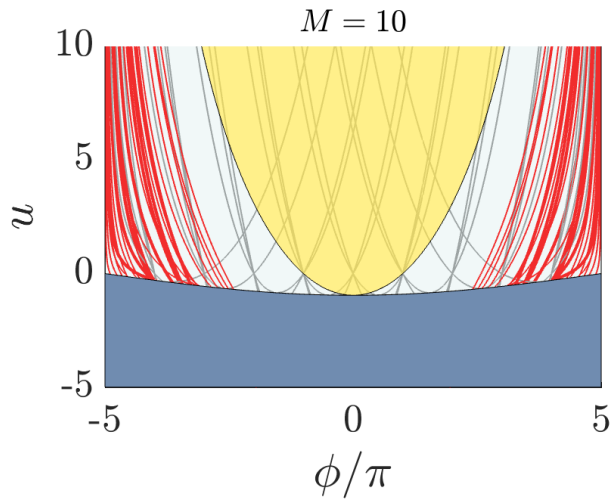
$$\omega_{q_1} + \omega_{q_2} - \omega_{q_1+q_2} = 0 \qquad \text{[gray lines]}$$

$$\omega_{q_1} + \omega_{q_2} + \omega_{-q_1-q_2} = 0 \qquad \text{[red lines]}$$









Considering the "1:2" resonance for the $m = 1$ flow-state of the $M = 4$ ring, setting $q_1 = q_2 = q = 2\pi/4$, we get from $2\omega_q + \omega_{-2q} = 0$ the resonance condition

$$u \;=\; 4\cot\left(\frac{\Phi}{4}\right)\left[3\cos\left(\frac{\Phi}{4}\right) - \sqrt{6 + 2\cos\left(\frac{\Phi}{2}\right)}\right]$$

Metastability - The Big Picture

- (traditional) Energetic metastability, aka Landau criterion.
- (traditional) Dynamical metastability via linear stability analysis, aka BdG.
- Strict dynamical metastability (KAM, applies if $d = 2$)
- Quasi dynamical metastability (might be the case for $d > 2$)

In the absence of constants of motion, a generic system with $d > 2$ degrees-of-freedom tends to be ergodic. But the equilibration might be an extremely slow process.

Quasi stability might become Quantum stability due to dynamical localization.
The breaktime is determined from the breakdown of the QCC condition.