

Non-perturbative response: chaos versus disorder

Doron Cohen¹ and Tsampikos Kottos²

¹ Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel

² Max-Planck-Institut für Strömungsforschung, Bunsenstr. 10, D-37073 Göttingen, Germany

Received 22 May 2003, in final form 30 July 2003

Published 24 September 2003

Online at stacks.iop.org/JPhysA/36/10151

Abstract

Quantized chaotic systems are generically characterized by two energy scales: the mean level spacing Δ , and the bandwidth $\Delta_b \propto \hbar$. This implies that with respect to driving such systems have an adiabatic, a perturbative and a non-perturbative regimes. A ‘strong’ nonlinearity in the response, due to a quantal non-perturbative effect, is found for *disordered* systems that are described by random matrix theory models. Is there a similar effect for quantized *chaotic* systems? Theoretical arguments cannot exclude the existence of an analogous ‘weak’ version of the above-mentioned nonlinear response effect, but our numerics demonstrates an unexpected degree of semiclassical correspondence.

PACS number: 05.45.Mt

1. Introduction

1.1. The two energy scales in quantum chaos

The name ‘quantum mechanics’ is associated with the idea that the energy is quantized. For generic (chaotic) system the mean level spacing is $\Delta \propto \hbar^d$, where d is the dimensionality of the system. However, one should recognize that there is a second energy scale $\Delta_b \propto \hbar$ which is introduced by quantum mechanics. This \hbar energy scale is related to the chaos implied decay of the classical correlations. It is known in the literature as the ‘non-universal’ energy scale [1], or as the ‘bandwidth’ [2]. The dimensionless bandwidth is defined as $b = \Delta_b/\Delta$. For reasonably small \hbar one has $b \gg 1$.

This observation, of having two energy scales, has motivated the study of Wigner model [3] within the framework of random matrix theory (RMT). This model, which is defined in terms of Δ and Δ_b , is totally artificial: it does not possess any classical limit. Still note that it can be reinterpreted as a model for the motion of a particle in a quasi one-dimensional *disordered* lattice [4].

The main focus of quantum chaos studies (so far) was on issues of spectral statistics [5]. In that context it turns out that the sub- \hbar statistical features of the energy spectrum are ‘universal’, and obey the predictions of random matrix theory. Non-universal (system specific) features

are reflected only in the large scale properties of the spectrum (analysing energy intervals $> \Delta_b$).

1.2. Regimes in the theory of driven systems

In recent years we have made some progress in understanding the theory of driven quantized chaotic systems [6–9]. Driven systems are described by Hamiltonian $\mathcal{H}(Q, P, x(t))$, where $x(t)$ is a time-dependent parameter. Such systems are of interest in mesoscopic physics (quantum dots), as well as in nuclear, atomic and molecular physics. The time-dependent parameter $x(t)$ may have the significance of external electric field or magnetic flux or gate voltage. Linear driving $x(t) = Vt$ is characterized by one parameter (V), while more generally a periodic driving $x(t) = Af(t)$ is characterized by both amplitude (A) and frequency (Ω). Due to the time dependence of $x(t)$, the energy of the system is not a constant of motion. Rather the system makes ‘transitions’ between energy levels, and therefore absorbs energy.

The main object of our studies is the energy spreading kernel $P_t(n|m)$. Regarded as a function of the level index n , it gives the energy distribution after time t , where m is the initial level. Having two quantal energy scales (Δ, Δ_b) implies the existence of different quantum mechanical (QM) V regimes [6, 7], or more generally (A, Ω) regimes [8], in the theory of $P_t(n|m)$. Most familiar is the QM adiabatic regime (very small V), whose existence is associated with having finite Δ . Outside of the adiabatic regime we are used to the idea that there is a perturbative regime, where the Fermi golden rule applies, leading to a Markovian picture of the dynamics, with well-defined transition rates between levels. Less familiar [6–8] is the QM non-perturbative regime (V is quantum mechanically large, but still classically small) whose existence is associated with the energy scale Δ_b . As implied by the terminology, in the QM non-perturbative regime perturbation theory (to any order) is not a valid tool for the analysis of the energy spreading. Consequently, the Fermi golden rule picture of the dynamics does not apply there.

1.3. Linear response theory

Of special importance (see the discussion below) is the theory for the variance $\delta E(t)^2 = \sum_n P_t(n|m)(E_n - E_m)^2$ of the energy spreading. Having $\delta E(t) \propto A$ means *linear response*. If $\delta E(t)/A$ depends on A we call it ‘nonlinear response’. In this paragraph we explain that linear response theory (LRT) is based on the ‘LRT formula’

$$\delta E(t)^2 = A^2 \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{F}_t(\omega) \tilde{C}(\omega). \quad (1)$$

Two spectral functions are involved: one is the power spectrum $\tilde{C}(\omega)$ of the fluctuations, and the other $\tilde{F}_t(\omega)$ is the spectral content of the driving. See equations (4) and (5) for exact definitions. A special case of equation (1) is the sudden limit ($V = \infty$) for which $f(t)$ is a step function, hence $F_t(\omega) = 1$, and accordingly

$$\delta E = \sqrt{C(0)} \times A \quad (\text{‘sudden’ case}). \quad (2)$$

Another special case is the response for persistent (either linear or periodic) driving. In such case the long time limit of $F_t(\omega)$ is linear in time (e.g. for linear driving ($f(t) = t$) we get $F_t(\omega) = t \times 2\pi\delta(\omega)$). This implies diffusive behaviour:

$$\delta E(t) = \sqrt{2D_E t} \quad (\text{‘Kubo’ case}). \quad (3)$$

In the latter case the expression for $D_E \propto A^2$ is known as Kubo (or Kubo–Greenwood) formula, leading to a fluctuation–dissipation relation [9].

The LRT formula equation (1) has a simple classical derivation³. From now on it goes without saying that we assume that the *classical* conditions on (A, Ω) for the validity of equation (1) are satisfied (no \hbar involved in such conditions). The question is *what happens to the validity of LRT once we ‘quantize’ the system*. Can we trust equation (1) for any (A, Ω) ? Or maybe we can trust it only in a restricted regime? In previous publications [6–9], we were able to argue the following:

- (A) The LRT formula can be trusted in the perturbative regime, with the exclusion of the adiabatic regime.
- (B) In the sudden limit the LRT formula can be trusted also in the non-perturbative regime.
- (C) In general, the LRT formula cannot be trusted in the non-perturbative regime.
- (D) The LRT formula can be trusted deep in the non-perturbative regime, provided the system has a classical limit.

For a system that does not have a classical limit (Wigner model) we were able to demonstrate [8] that LRT fails in the non-perturbative regime. Namely, for Wigner model the response $\delta E(t)/A$ becomes A dependent for large A , meaning that the response is nonlinear. Hence the statement in item (C) above has been established. We had argued that the observed nonlinear response is the result of a quantal non-perturbative effect. *Do we have a similar type of nonlinear response in the case of quantized chaotic systems?* The statement in item (D) above seems to suggest that the observation of such nonlinearity is not likely. Still, we argue below that this does not exclude the possibility of observing a ‘weak’ nonlinearity.

2. Perturbation theory and linear response

The immediate (naive) tendency is to regard LRT as the outcome of first-order perturbation theory (FOPT). In fact the regimes of validity of FOPT and of LRT do not coincide. On the one hand we have the adiabatic regime where FOPT is valid as a leading order description, but not for response calculation (see further details below). On the other hand, the validity of equation (1) goes well beyond FOPT. This leads to the (correct) identification [6–8] of what we call the ‘perturbative regime’. The border of this regime (in (A, Ω) space) is determined by the energy scale Δ_b , while Δ is not involved. Outside of the perturbative regime we cannot trust the LRT formula. However, as we further explain below, the fact that equation (1) is not valid in the non-perturbative regime, does not imply that it *fails* there.

We stress again that one should distinguish between ‘non-perturbative response’ and ‘nonlinear response’. These are not synonyms. As we explain in the next paragraph, the adiabatic regime is ‘perturbative’ but ‘nonlinear’, while the semiclassical limit is ‘non-perturbative’ but ‘linear’.

In the *adiabatic regime*, FOPT implies zero probability to make a transition to other levels. Therefore, to the extent that we can trust the adiabatic approximation, all the probability remains concentrated in the initial level. Thus, in the adiabatic regime, equation (1) is not a valid formula: it is essential to use higher orders of perturbation theory, and possibly non-perturbative corrections (Landau–Zener [10]), in order to calculate the response. Still, FOPT provides a meaningful leading order description of the dynamics (i.e. having no transitions), and therefore we do not regard the adiabatic nonlinear regime as ‘non-perturbative’.

³ For an elementary classical and quantum mechanical derivations of equation (1), using either fixed or adiabatic basis see Appendices C and D of [7].

In the *non-perturbative regime* the evolution of $P_t(n|m)$ cannot be extracted from perturbation theory: not in leading order, neither in any order. Still it does not necessarily imply a nonlinear response. In contrast, the semiclassical limit is contained in the (deep) non-perturbative regime [8]. There, the LRT formula equation (1) is in fact valid. But its validity is *not* a consequence of perturbation theory, but rather the consequence of *quantal-classical correspondence*.

3. The quest for non-perturbative response

As stated above, an effect of nonlinear response due to the *quantum mechanical* non-perturbative nature of the dynamics, has been demonstrated so far only for Wigner model [8]. There, its existence is related to the *disordered* RMT nature of the model (see the discussion below). Semiclassical correspondence considerations seem to exclude the manifestation of this disorder-related nonlinearity in the case of quantized *chaotic* systems. In this paper, we explain that this does not exclude the possibility of having a ‘weak’ version of this effect. We also report the results of an intense numerical effort aimed at finding a ‘weak’ nonlinearity in the case of a simple low-dimensional quantized chaotic systems. To our surprise, an unexpected degree of semiclassical correspondence is observed.

It is appropriate here to clarify the notions of ‘weak’ and ‘strong’ effects. In the literature regarding the dynamics in disordered lattices one distinguishes between ‘weak’ and ‘strong’ localization effects. The former term implies that while the leading behaviour is classical (diffusion), there are ‘on top’ quantum mechanical corrections (enhanced return probability). In contrast to that the term ‘strong’ implies that the classical description fails even as a leading order description. In the literature regarding quantum chaos we have the effect of ‘scarring’, which should be regarded as ‘weak’ effect. ‘Strong’ quantum mechanical effects (e.g. dynamical localization in 1D kicked systems [13]) are non-generic: the leading order behaviour of generic quantized chaotic systems is typically classical. In the present context of driven systems, we use the terms ‘weak’ and ‘strong’ in the same sense: the adjective ‘weak’ is associated with the (conjectured) nonlinear response of quantized driven chaotic systems, while the adjective ‘strong’ is associated with the (established) nonlinear response in the corresponding RMT (Wigner) model.

4. The numerical findings

How do we detect nonlinear response? The most straightforward way is to fix the pulse shape $f(t)$ and to plot $\delta E/A$ versus A . A deviation from constant value means ‘nonlinear’ response. The simulations below are performed for a quantized chaotic system. Due to obvious numerical limitations we will consider the response to one-pulse driving, rather than to persistent (periodic) driving. *The central question is whether the observed nonlinear effect is of semiclassical origin, or of novel quantum mechanical origin.* We deal with this issue below.

In our numerical simulations (figure 1) we have considered a particle in a two-dimensional ($d = 2$) anharmonic well. This model (with deformation parameter $x = \text{const}$) is defined in [11, 12]. In the energy region of interest ($E \sim 3$), the classical motion inside the two-dimensional well (2DW) is chaotic, with characteristic correlation time $\tau_{cl} \sim 1$. For the following presentation it is enough to say that the quantum mechanical Hamiltonian is represented by a matrix $\mathcal{H} = \mathbf{E} + x(t)\mathbf{B}$, where \mathbf{E} is a diagonal matrix with mean level spacing $\Delta \approx 4.3 \times \hbar^d$, and \mathbf{B} is a banded matrix. The bandwidth (in energy units) is $\Delta_b = 2\pi\hbar/\tau_{cl}$.

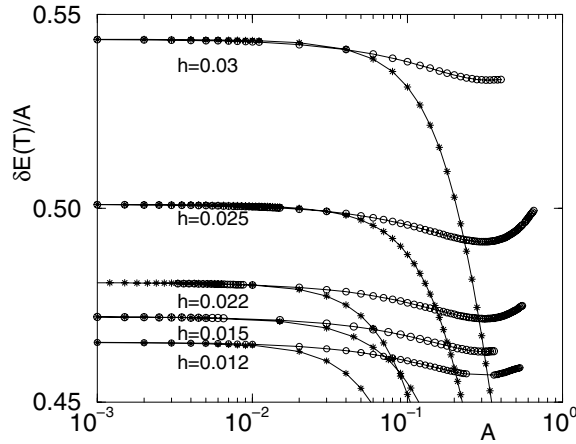


Figure 1. The response $\delta E/A$ as a result of a rectangular pulse ($T = 0.375$). Deviation from $\delta E/A = \text{const}$ implies nonlinear response. All the data are averaged over a number of different initial conditions. The simulations are performed with the 2DW Hamiltonian (circles), and also with the associated RMT model (stars). See text for explanations.

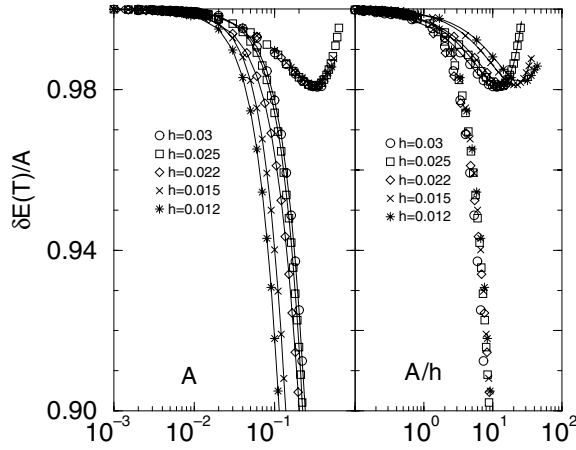


Figure 2. Scaled versions of figure 1. The vertical scaling is aimed at removing the weak \hbar dependence of the bandprofile. In (b) the horizontal scaling is aimed at checking whether the deviation from linear response is in fact a quantal non-perturbative effect.

The bandprofile (see figure 2 of [10]) is described by a spectral function which is defined as follows:

$$\tilde{C}(\omega) = \sum_{n(\neq m)} |\mathbf{B}_{nm}|^2 2\pi \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right) \quad (4)$$

with implicit average over the reference state m . The bandprofile, as defined above, can be determined from the classical dynamics. This means that $\tilde{C}(\omega) \approx \tilde{C}^{cl}(\omega)$ where $\tilde{C}^{cl}(\omega)$ is the Fourier transform of a classical correlation function $C^{cl}(\tau)$. The \hbar dependence of $\tilde{C}(\omega)$ is relatively weak.

The driving pulse in our numerical simulations has a rectangular shape. This means $f(0) = f(T) = 0$ and $f(0 < t < T) = A$, where $T = 0.375$. The spectral content of the

driving is defined as

$$\tilde{F}_t(\omega) = \left| \int_0^t \dot{f}(t') e^{i\omega t'} dt' \right|^2. \quad (5)$$

The spectral content of the driving after a rectangular pulse is $F_t(\omega) = |1 - e^{i\omega T}|^2$. We have also made simulations (not presented) with a driving scheme that involves a positive pulse $+A$ followed by a negative pulse $-A$, with the intention of considering eventually a persistent (multi-cycle) periodic driving. However, we have realized that all the relevant physics is observed already in the single pulse case. Note that the regime diagram for either linear or (as in the following simulations) rectangular driving pulse, is greatly simplified, because the driving is characterized by only one parameter (V in the former case, A in the latter case).

Let us look carefully at the results of the 2DW simulations (figure 1). For small A we see as expected ‘linear response’ meaning $\delta E/A = \text{const}$, as implied by equation (1). Note that the ‘constant’ has a weak \hbar dependence (a 10% effect). This is due to the above-mentioned weak dependence of $\tilde{C}(\omega)$ on \hbar . So this quantum mechanical effect is quite trivial, and has a simple explanation within LRT. Now let us look what happens for large A . We clearly see a 2% deviation from linear response. In what follows we discuss the reason for this nonlinear effect.

For the sake of comparison we also perform simulations with an effective RMT model that corresponds to the 2DW model Hamiltonian. The effective RMT model is obtained by randomizing the signs of the off-diagonal elements of the \mathbf{B} matrix. The effective RMT Hamiltonian has the *same* bandprofile $\tilde{C}(\omega)$ as the original (2DW) Hamiltonian. Therefore, as far as LRT equation (1) is concerned, the response should be the *same*. Still we see that at the same A regime, as in the case of the 2DW simulations, we have deviation from linear response. However, this nonlinear deviation is much stronger.

Looking at the curves of figure 1, it is very tempting to regard the observed nonlinear 2% effect in the 2DW simulations as a ‘weak’ version of the ‘strong’ effect which is observed in the corresponding RMT simulations. However, the careful analysis below indicates that apparently this is not the case.

5. Discussion and analysis

In analysing the validity of the LRT formula, it is instructive to consider first the sudden limit equation (1). This limit has been studied in [11]. The spreading profile $P(n|m)$, after the sudden change in x , depends on the amplitude A of the perturbation. (We omit the time index t , which is of no relevance in this limit.) The perturbative regime is $A < A_{\text{prt}}$, where $A_{\text{prt}} = 2\pi\hbar/(\tau_{cl}\sqrt{C(0)})$. For the 2DW simulations $A_{\text{prt}} = 5.3 \times \hbar$. In the perturbative regime $P(n|m)$ has a core-tail structure (the generalization of Wigner Lorentzian), and the variance δE^2 is determined by the first-order tail component of the energy distribution. For $A > A_{\text{prt}}$ the spreading profile $P(n|m)$ becomes non-perturbative. This means that the perturbative tail (if survives) is no longer the predominant feature. Thus the variance is determined by the non-perturbative component (the ‘core’) of the energy distribution. The remarkable fact is that the crossover from the perturbative A regime to the non-perturbative A regime is *not* reflected in the variance (see figure 5 of [11]). The agreement with equation (2) is perfect. Taking into account the ‘dramatic’ differences in the appearance of $P(n|m)$, this looks quite surprising. In fact (see section 12 of [11]) there is a simple proof⁴ that equation (2) remains exact beyond any order of perturbation theory, which means that it is exact even in the non-perturbative regime where perturbation theory is not applicable.

⁴ We thank F M Izrailev for pointing out the existence of this proof in [3].

We turn back to our simulations, where we have a rectangular pulse (rather than step function). Here the sudden limit does not apply, and the dynamics within the time interval $0 < t < T$ should be taken into account. If we take an eigenstate of \mathbf{E} and propagate it using $\mathbf{E} + \mathbf{A}\mathbf{B}$, then we get in the classical case *ballistic* spreading followed by saturation. ('Eigenstate' in the classical case means microcanonical distribution.) This is true for any A . Quantum mechanically we observe in the 2DW model simulations a similar ballistic behaviour [12], whereas in the corresponding RMT model there is an intermediate stage of *diffusion* [12]. This diffusion is of non-perturbative nature, and it is related to the 'disorder' which is artificially introduced via the sign-randomization procedure. The strong nonlinear response effect [8] is a consequence of this diffusion.

Coming back to the 2DW model, we realize that there is no 'disorder' built into the model, and therefore no diffusion. Still, looking at figure 1, it is tempting to interpret the observed 2% nonlinear deviation as a 'weak' version of the strong nonlinear effect. Moreover, regarded as such, it vanishes, as expected, in the deep non-perturbative regime, which had been argued on the basis of semiclassical correspondence considerations [8].

In order to properly determine whether the dips in figure 1 are the result of the QM non-perturbative nature of the dynamics, we have rescaled the vertical axis, and plotted the response once (figure 2(a)) versus A , and once (figure 2(b)) versus A/\hbar . On the basis of the scaling we see that the strong nonlinear response in the RMT case is indeed the result of the quantal (\hbar -dependent) non-perturbative effect. In contrast to that the \hbar -independent scaling in the 2DW case, indicates that the nonlinear deviation there is of 'semiclassical' rather than of 'quantal non-perturbative' nature.

6. Conclusion

In conclusion, theoretical arguments cannot exclude the existence of a 'weak' nonlinearity in the response of a driven quantized chaotic system, which is due to a quantum mechanical non-perturbative effect. But our careful numerics, regarding a simple low-dimensional system, demonstrates an unexpected degree of semiclassical correspondence. Our findings should be regarded as the outcome of an ongoing quest, which has not ended, and that aims to find novel quantum mechanical deviations from linear response theory.

Acknowledgments

This work was supported by a grant from the GIF, the German-Israeli Foundation for Scientific Research and Development and by the Israel Science Foundation (grant no 11/02).

References

- [1] Berry M V 1991 *Chaos and Quantum Systems* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: Elsevier)
- [2] Feingold M and Peres A 1986 *Phys. Rev. A* **34** 591
 Feingold M, Leitner D and Wilkinson M 1991 *Phys. Rev. Lett.* **66** 986
 Feingold M, Gioletta A, Izrailev F M and Molinari L 1993 *Phys. Rev. Lett.* **70** 2936
 Wilkinson M, Feingold M and Leitner D 1991 *J. Phys. A: Math. Gen.* **24** 175
 Fyodorov Y V and Mirlin A D 1991 *Phys. Rev. Lett.* **67** 2405
- [3] Wigner E 1955 *Ann. Math.* **62** 548
 Wigner E 1957 *Ann. Math.* **65** 203
 Flambaum V V, Gribakina A A, Gribakin G F and Kozlov M G 1994 *Phys. Rev. A* **50** 267
 Casati G, Chirikov B V, Guarneri I and Izrailev F M 1993 *Phys. Rev. E* **48** R1613

- Casati G, Chirikov B V, Guarneri I and Izrailev F M 1996 *Phys. Lett. A* **223** 430
Cohen D, Izrailev F and Kottos T 2000 *Phys. Rev. Lett.* **84** 2052
Fyodorov Y V, Chubykalo O A, Izrailev F M and Casati G 1996 *Phys. Rev. Lett.* **76** 1603
- [4] Fyodorov Y V and Mirlin A D 1994 *Int. J. Mod. Phys. B* **8** 3795
[5] Haake F 2000 *Quantum Signatures of Chaos* (Berlin: Springer)
Stockmann H J 1999 *Quantum Chaos: An Introduction* (Cambridge: Cambridge University Press)
- [6] Cohen D 1999 *Phys. Rev. Lett.* **82** 4951
Cohen D 2000 *Ann. Phys., NY* **283** 175
[7] Cohen D and Wisniacki D A 2003 *Phys. Rev. E* **67** 026206 (*Preprint nlin.CD/0204025*)
[8] Cohen D and Kottos T 2000 *Phys. Rev. Lett.* **85** 4839
[9] For an updated review see Cohen D 2002 *Proc. 38th Karpacz Winter School of Theoretical Physics: Dynamics of Dissipation* ed P Garbaczewski and R Olkiewicz (Berlin: Springer) <http://link.springer.de/link/service/series/2669/tocs/t2597.htm>
See also Cohen D 2000 *Proc. Int. School of Physics Enrico Fermi Course CXLIII* ed G Casati, I Guarneri and U Smilansky (Amsterdam: IOS Press)
- [10] Wilkinson M 1988 *J. Phys. A: Math. Gen.* **21** 4021
Wilkinson M 1987 *J. Phys. A: Math. Gen.* **20** 2415
[11] Cohen D and Kottos T 2001 *Phys. Rev. E* **63** 36203
[12] Kottos T and Cohen D 2001 *Phys. Rev. E* **64** R-065202
[13] For review and references see Fishman S 1991 *Quantum chaos Proc. Int. School of Physics 'Enrico Fermi', Course CXIX* ed G Casati, I Guarneri and U Smilansky (Amsterdam: North-Holland)