Localization, Dynamical Correlations, and the Effect of Colored Noise on Coherence

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Localization in the quantum-kicked-rotator problem leads to nontrivial dynamical correlations that are absent in the classical limit. Consequently, both coherence and diffusion in the presence of colored noise depend on its long-range autocorrelations. This implies that a Markovian treatment of the dynamics for a system that is coupled to a low-temperature heat bath is not valid even if the system is classically chaotic.

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The quantum-kicked-rotator (QKR) problem [1] constitutes a prototype example for the suppression of classical chaos due to quantal localization [2]. It is closely related to the studies of Zener dynamics in multilevel systems [3] and the ionization of hydrogen atoms by a microwave electric field [4]. It was found by Ott, Antonsen, and Hanson [5] that uncorrelated white noise destroys coherence and hence localization. However, if the noise arises from a coupling to a heat bath then a more detailed investigation is desired. It should take into account two ingredients. One is *noise autocorrelations* which are expected at low temperature [6]. The other is *friction* which may result in dissipation of energy.

The combined effect of noise and dissipation has been investigated by Dittrich and Graham [7]. Lately it has been shown [8] that their model is non-Ohmic in the Caldeira-Leggett sense [9]. Consequently two other different models have been introduced [8,10] where the quantum kicked rotator has been assumed to be coupled to an Ohmic bath either via its momentum variable [10] or via its position coordinate [8]. In the case of the former coupling scheme [10] friction does not affect significantly the long-time dynamics of the QKR and therefore may be ignored [10]. Coupling via the position variable [8] results in damping that is proportional to velocity and hence to dissipation of energy. In the latter case, if the coupling is weak, the coherence time is much shorter than the relaxation time even in the limit of zero temperature [8]. Consequently a phenomenological Fokker-Plank treatment of the relaxation process is sufficient, and friction may be ignored for the purpose of determination of the diffusion coefficient.

By inspection of the Feynman-Vernon [11] formalism it has been shown [6,8,10,11] that if friction is ignored then the heat bath has the same effect as that of a stochastic force ("noise"). In the case of an Ohmic bath the noise is white at high temperatures. At low temperatures, due to the quantum nature of the bath, it possesses long-range (negative) autocorrelations [6]. These noise autocorrelations that arise at low temperatures affect significantly the time evolution of *integrable* systems [6]. For example, in the case of either an undriven particle [6,12] or an undriven rotator [10] these negative autocorrelations result in suppression of linear diffusion (variance grows linearly with time) and instead a logarithmic behavior (variance grows logarithmically with time) is found. A *Markovian treatment* of the dynamics, which underlies the frequently used master-equation approach [13], ignores these noise autocorrelations [6]. If the system is treated classically and is known to be *chaotic* then this should not be important—due to the exponential instability of the phase-space trajectories we expect no *memory* for long-range autocorrelations. However, if the dynamics is treated within the framework of quantum mechanics, then one may expect a manifestation of long-range *dynamical correlations*.

In this Letter we argue the following: (a) Both diffusion and coherence depend on the noise autocorrelations. (b) This dependence is due to nontrivial dynamical correlations that are absent in the classical limit. (c) The dynamical correlations are related to the crossover from diffusion to localization. (d) Mott's resonant-state picture of Anderson localization is applicable to the investigation of these long-range correlations.

The time evolution of the QKR is given by iterations with the one-step propagator

$$\hat{U}_0=\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2\right], \qquad (1)$$

with $[\hat{x},\hat{p}]=i\hbar$ and periodic boundary conditions on $[0,2\pi]$ are imposed. Fishman, Grempel, and Prange [2] have argued that the eigenstates $|r\rangle$ of the one-step propagator $\hat{U}_0$ are localized in the p representation with localization length $\hbar\xi$ which is given [14] for large K by $\xi\approx\frac{1}{4}K^2/\hbar^2$ (we shall assume from now on that $1\ll K$). The eigenvalues of $\hat{U}_0$ will be denoted by $e^{-i\omega_r}$, where ω_r are the quasienergies. The differences $\omega_r-\omega_s$ of the latter will be denoted by ω_{rs} with the convention $-\pi<\omega_{rs}<\pi$. The localization of the eigenstates in the present QKR problem is similar to Anderson localization in solid-state physics. Lately, strong evidence has been presented that Mott's resonant-state picture [15] of Anderson localization is applicable also to this localization problem [16]. It follows that not all the eigenstates have a simple localization structure; in particular, one expects to find pairs of almost-degenerate double-hump states. Let $|r\rangle$ and $|s\rangle$ be a particular pair—the dipole matrix element $|\langle s|\hat{p}|r\rangle|$ is then exceptionally large, approximately $\frac{1}{2}\hbar\xi\ln(\Delta/|\omega_{rs}|)$, where Δ is of the order $1/\xi$. The spectral density of these pairs is ξ/π in the small frequency interval $|\omega|<\Delta$, and zero elsewhere [17].

We consider here a rotator that is coupled to an external c-number noise source. The one-step propagator is $\exp[-(i/\hbar)\mathcal{H}_{\text{int}}]U_0$. A linear coupling scheme is assumed, namely, $\mathcal{H}_{\text{int}}=f_t\hat{X}$, where $\hat{X}$ is a dynamical variable defined subsequently. The dynamical behavior should be averaged over realizations of the sequences f_t such that $\langle f_t f_{t'}\rangle=\nu(t-t')$. Explicit expressions for $\nu(\tau)$ in the case of an Ohmic bath are presented in Refs. [6,8,10]. In the limit of zero temperature the total area under $\nu(\tau)$ goes to zero but the noise does not vanish, instead it has negative autocorrelations such that $\nu(\tau)\propto-1/\tau^2$ for $1<\tau$. Particular choices for the dynamical variable $\hat{X}$ are $\hat{X}=p$ (Ref. [10]) and $\hat{X}=\sin x$ (Ref. [17]).

The average decay rate Γ of a quasienergy eigenstate plays a central role in the theory that is presented in this Letter. In order to find Γ one estimates the transition rates between localized eigenstates using a leading-order perturbative calculation, and averaging over noise realizations. Then, one has to sum over the final states and average over the initial states. Such a calculation is presented in detail in Refs. [8,17]. The final result can be cast into the form

$$\Gamma=\frac{1}{\hbar^2}\sum_{\tau=-\infty}^{\infty}C_X(\tau)\nu(\tau)\,, \tag{2}$$

where $\nu(\tau)$ is the noise-autocorrelation function, $\nu\equiv\nu(0)$ is its variance, and $C_X(\tau)$ is the Fourier transform of the spectral function

$$C_X(\omega)\equiv\lim_{\mathcal{N}\to\infty}\frac{1}{\mathcal{N}}\sum_r\sum_{s(\neq r)}|\langle s|\hat{X}|r\rangle|^2 2\pi\delta(\omega-\omega_{sr})\,. \tag{3}$$

We use $\mathcal{N}$ to denote the dimension of Hilbert space. A natural way to define the coherence time is $t_c\equiv\Gamma^{-1}$. Following Ott, Antonsen, and Hanson [5] it is argued that for weak noise the diffusion process in momentum space is similar to a random walk on a grid with spacing $\hbar\xi$ and hopping probability Γ. The diffusion coefficient is then

$$\mathcal{D}\approx(\hbar\xi)^2\Gamma\,. \tag{4}$$

Further study to establish the validity of this formula which is based on short-time perturbative considerations is presented elsewhere [17], and an analytical determination of prefactor of order unity which is not fixed by the present heuristic approach is presented there [17]. *It follows from (2) and (4) that both coherence and diffusion depend on the noise autocorrelations provided $C_X(\tau)$ has nontrivial structure.* We therefore turn to study in detail two particular couplings with the corresponding correlation functions $C_p(\tau)$ and $C_s(\tau)\equiv C_{\sin x}(\tau)$. Both functions are related to the dynamical crossover from diffusion to localization in the *absence* of noise. This relation enables one to get insight of their global behavior and furthermore to extract them numerically.

In order to quantify the dynamical crossover one may define the dispersion function $E(t)\equiv\langle\langle[\hat{p}(t)-\hat{p}(0)]^2\rangle\rangle$. The notation $\langle\langle\hat{O}\rangle\rangle\equiv(1/\mathcal{N})\text{trace}(\hat{O})$ stands for quantum statistical average. It corresponds in the semiclassical limit to uniform average over phase-space cells. In order to obtain $E(t)$ numerically one averages the dispersion $\langle\psi(t)|(\hat{p}-p)^2|\psi(t)\rangle$ over the initial preparation $|\psi(0)\rangle\equiv|p\rangle$ with $p=\hbar n$ $(n=0,\pm1,\pm2,\dots)$. The time derivative of $E(t)$ will be denoted by $D(t)$, namely, $D(t)\equiv E(t+1)-E(t)$. We found that the crossover has the following functional form:

$$D(t)=\begin{cases}D_0e^{-t/t^*} & \text{for } t<\mathcal{O}(t^*)\,,\\ cD_0(t^*/t)^{1+\beta} & \text{for } \mathcal{O}(t^*)<t\,,\end{cases} \tag{5}$$

with $D_0\approx\frac{1}{2}K^2$, $t^*\approx2\xi$, $\beta\approx0.75$, and $c\approx0.5$. More details including analytical considerations are presented elsewhere [17]. It should be noted that the asymptotic power-law behavior of the crossover has already been reported by Berman and Izrailev [18,19]. They have tried to relate β to spectral properties of the system using a heuristic picture [18]. Further study [19] has revealed that in order to recover (formally) their result one should assume that local level statistics is a well-defined notion. In view of the later work by Dittrich and Smilansky [16] this ansatz does not hold. A different strategy will therefore be reported here. It predicts $\beta=1$ in agreement with a later phenomenological argument due to Chirikov [20].

The correlation function $C_p(\tau)$ is related to the crossover via

$$E(t)=2[C_p(0)-C_p(t)]\,. \tag{6}$$

In the case where $C_s(\tau)\equiv\langle\langle\sin\hat{x}(\tau)\sin\hat{x}(0)\rangle\rangle$ one finds the relation [17]

$$D(t)\equiv K^2\sum_{\tau=-t}^{t}C_s(\tau)\,. \tag{7}$$

The first few sine correlations, namely, $C_s(0)=\frac{1}{2}$, $C_s(1)=0$, $C_s(2)$, $C_s(3)$, and $C_s(4)$ have been computed by Shepelyansky [14], and it was found that a good approximation is to use the classical expressions [21] with K replaced by $(2/\hbar)\sin(\hbar/2)K$. Classically the other correlations are very small and the sum rule

$$K^2\sum_{\tau=-\infty}^{\infty}C_s(\tau)=D_0$$

is satisfied. This sum is dominated by the first few terms. Quantum mechanically the diffusion is (asymptotically) zero, and therefore $\sum_{\tau=-\infty}^{\infty}C_s(\tau)=0$. Thus $C_s(\tau)$ has a nonclassical negative tail that compensates exactly the short-range contribution. On the basis of (6) and (7) and upon using (5) it follows that *dynamical correlations in*

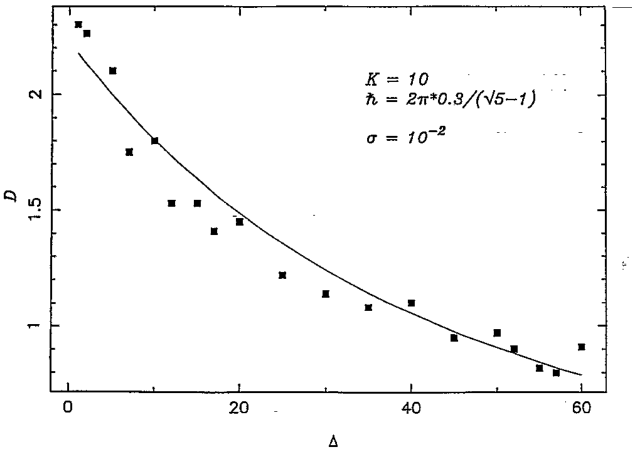

FIG. 1. The diffusion coefficient $\mathcal{D}$ as a function of Δ. See text for details. An average over 100 initial conditions and noise realizations has been performed in order to determine $\mathcal{D}$ on the basis of simulations. The curve represents an analytical estimate with no fitting parameters.

the QKR model decay exponentially on the time scale t^* while on a larger time scale a slower power-law decay manifests itself. The power-law behavior in the case of $C_p(\tau)\sim 1/t^\beta$ indicates that $C_p(\omega)$ is singular in the vicinity of $\omega=0$. This implies, by inspection of (3), exceptionally large dipole matrix elements. Mott's picture enables one to calculate the contribution of the pairs of double-hump states to the spectral function. Using the estimates for their dipole matrix element and spectral density, one obtains $C_p(\omega)|_{\rm res}=\frac{1}{2}\hbar^2\xi^3\ln^2|\Delta/\omega|$ for $|\omega|<\Delta$ and zero for $\Delta<|\omega|$. The Fourier transform can be calculated analytically. It leads to the asymptotic behavior

$$C_p(\tau)\sim\tfrac{1}{2}\hbar^2\xi^3(1/\tau)\ln|\Delta\tau| \quad \text{for } \xi\ll|\tau|\,. \qquad (8)$$

This formula agrees with the power-law behavior in (5) provided $c=\frac{1}{8}\ln(t/t^*)$ and $\beta=1$. The deviation of the numerical results from this prediction may be due to a very long transient behavior. Another possibility is that it is due to the simplified nature of Mott's picture. Namely, most of the eigenstates possess structures that are neither simple exponentials nor distinguished double humps.

To test the predictions of Eq. (4) with (2) some numerical experiments have been performed [17]. Figure 1 illustrates the results of such a typical experiment. The coupling to the noise source is $\mathcal{H}_{\rm int}=f_t\sin\hat{x}$ and the noise autocorrelation function is $\nu(t-t')=\sigma\max[1-|(t-t')/\Delta|,0]$. The diffusion coefficient $\mathcal{D}$ has been measured as a function of Δ. *Diffusion is found to be suppressed due to the positive noise autocorrelations.* The smooth curve represents the expected results on the basis of a refined version of Eq. (4) which does not involve an undetermined prefactor. The decay rate has been obtained by applying (2) where the correlation function $C_s(\tau)$ had been determined via (7) and (5). There are no fitting parameters.

We also verified [17] that in the case of coupling via the momentum [10] the opposite effect is indeed realized, namely, *diffusion is suppressed due to negative noise autocorrelations.* This is expected from (4) with (2) where $C_p(\tau)$ is determined via (6). However, unlike the case of undriven rotator [10], even at zero temperature, in spite of the negative noise autocorrelations, diffusion does not vanish. The latter statement follows simply from the observation that due to the decay of dynamical correlations the decay rate (2) is always larger than zero.

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