Quantum Reversibility: Is there an Echo?

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We study the possibility to undo the quantum mechanical evolution in a time reversal experiment. The naive expectation, as reflected in the common terminology ("Loschmidt echo"), is that maximum compensation results if the reversed dynamics extends to the same time as the forward evolution. We challenge this belief and demonstrate that the time t_r for maximum return probability is in general shorter. We find that t_r depends on $\lambda = \varepsilon_{\mathrm{evol}}/\varepsilon_{\mathrm{prep}}$, being the ratio of the error in setting the parameters (fields) for the time-reversed evolution to the perturbation which is involved in the preparation process. Our results should be observable in spin-echo experiments where the dynamical irreversibility of quantum phases is measured.

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In this Letter we study the probability of return $P(t_1, t_2)$ for a generalized wave packet dynamics scenario. The system is prepared in some initial state Ψ_{prep}, which can be regarded as the outcome of a preparation procedure which is governed by a Hamiltonian $\mathcal{H}_{\mathrm{prep}}$. We assume that the quantum mechanical evolution is generated by Hamiltonians with a classically chaotic limit: The state is propagated for a time t_1 using a Hamiltonian $\mathcal{H}_1$, and then the evolution is time reversed for a time t_2 using a perturbed Hamiltonian $\mathcal{H}_2$. The corresponding evolution operators are U_1 and U_2. The probability of return to the initial state is

$$P(t_1, t_2) = |\langle\Psi_{\mathrm{prep}}|U_2(t_2)^{-1}U_1(t_1)|\Psi_{\mathrm{prep}}\rangle|^2. \tag{1}$$

There are two special cases that have been extensively studied in the literature. The traditional wave packet dynamics scenario [1] is obtained if we set $t_2 = 0$. In this context the "survival probability" is defined as

$$P_{\mathrm{SR}}(t) = P(t, 0). \tag{2}$$

The "Loschmidt echo" (LE) scenario is obtained if we set $t_1 = t_2 = t$. In this context the "fidelity" is defined as

$$P_{\mathrm{LE}}(t) = P(t, t). \tag{3}$$

The theory of the fidelity was the subject of intensive studies during the last 3 years [2–10]. It has been adopted as a standard measure for quantum reversibility following [2] and its study was further motivated by the realization that it is related to the analysis of dephasing in mesoscopic systems [11].

In the present Letter we consider the full scenario of a time reversal experiment. The probability to find the system in its original state is $P(t) = P(t, 0)$ before the time reversal ($t < T/2$), and

$$P(t) = P(T/2, t - T/2) \tag{4}$$

after the time reversal ($t > T/2$). The period T is the total time of the experiment. The naive expectation, which is also reflected in the term Loschmidt echo, is to have a maximum for $P(t)$ at the time $t = T$. We are going to show that this expectation is wrong. We find that the maximum return probability is obtained at a time t_r which in general is shorter than that. Namely,

$$T/2 \leq t_r \leq T. \tag{5}$$

If we have $t_r = T/2$ we say that there is no reversibility. If we have $t_r \sim T$ we say that we have a nearly perfect echo. We show that t_r/T is a function of a dimensionless parameter $0 < \lambda < \infty$. Namely,

$$\frac{t_r}{T} = f(\lambda); \qquad \lambda = \varepsilon_{\mathrm{evol}}/\varepsilon_{\mathrm{prep}}, \tag{6}$$

where $\varepsilon_{\mathrm{prep}}$ quantifies the difference between the evolution Hamiltonian $\mathcal{H}$ and the preparation Hamiltonian $\mathcal{H}_{\mathrm{prep}}$, while $\varepsilon_{\mathrm{evol}}$ quantifies the difference between the two instances $\mathcal{H}_1$ and $\mathcal{H}_2$ of the evolution Hamiltonian, which are used for the forward and for the time-reversed evolution, respectively. The idea is that there is no way to have complete control over the parameters (fields) of the systems. Therefore there is an unavoidable difference ($\varepsilon_{\mathrm{evol}}$) between these two instances of $\mathcal{H}$, which by the setup of the experiment are regarded as identical. The scaling function (6) takes the limiting value $f(0) = 1$ (echo) while for $\lambda > \lambda^*$ we get $f(\lambda) = 0.5$ (no reversibility). Here λ^* is some system-specific constant of order unity.

The most popular preparation which is considered in the literature, either in the context of wave packet dynamics or fidelity (LE) studies, is a Gaussian wave packet. Obviously the choice of such preparation is motivated mainly by the wishful thinking of theoreticians. However, in many applications, one is not so much interested in evolving an initial Gaussian wave packet. This is certainly the situation in quantum information processing

[12] and in spin-echo experiments [10] where one starts with a random initial state. Formally a Gaussian wave packet can be regarded as the ground state of a phase-space shifted Harmonic oscillator. Therefore it is characterized by a very large $\varepsilon_{\rm prep}$, leading to $\lambda \ll 1$. In this Letter we do not assume $\lambda \ll 1$, but rather consider the general case.

In order to develop a general theory we need a model in which we have control over both $\varepsilon_{\rm prep}$ and $\varepsilon_{\rm evol}$. We consider a quantized system whose classical analog has a positive Lyapunov exponent. Its Hamiltonian $\mathcal{H} = \mathcal{H}(Q, P; x)$ depends on a parameter (field variable) x which is determined by the experimental setup. For example it can be either a gate voltage or a magnetic flux. The dynamics takes place within a classically small (but quantum mechanically large) energy window. The classical dynamics is assumed to have a well-defined finite correlation time $\tau_{\rm cl}$. We consider classically small (but possibly quantum mechanically large) perturbations ($x \mapsto x + \delta x$). Accordingly, the Hamiltonian can be linearized as follows:

$$\mathcal{H} = \mathcal{E} + \delta x \mathcal{B}. \tag{7}$$

We define $\mathcal{H}_1$ and $\mathcal{H}_2$ by setting $\delta x = \pm \varepsilon_{\rm evol}$. The requirement of having classically small δx means that the phase-space structure of $\mathcal{H}_1$ and $\mathcal{H}_2$ is similar, and that any (small) difference in the chaoticity can be neglected (we have verified that this smallness condition is satisfied for the example below).

The preparation issue requires further discussion. The traditional possibility is to prepare a Gaussian wave packet (also known as a coherent state preparation). We regard this possibility as uncontrolled because the value of $\varepsilon_{\rm prep}$ is ill defined. To have a physically meaningful definition of $\varepsilon_{\rm prep}$ the natural procedure is as follows: We define a preparation Hamiltonian $\mathcal{H}_{\rm prep}$ by setting $\delta x = \varepsilon_{\rm prep}$. Then we start with an initial eigenfunction of $\mathcal{E}$ and evolve it with $\mathcal{H}_{\rm prep}$ until we get an ergodiclike steady state within an energy shell (note [13]). The width of this energy shell is proportional to $\varepsilon_{\rm prep}$. The resulting wave packet is used as an initial state for the time reversal experiment. For the purpose of comparison we shall consider also a Gaussian wave packet preparation. For such preparation $\lambda \ll 1$ irrespective of its energy width. The reason is that this preparation does not occupy ergodically its energy shell. Formally one can say that for a Gaussian wave packet $\mathcal{H}_{\rm prep}$ differs enormously from the evolution Hamiltonian. There is no point in quantifying this difference. This is the reason why we use our controlled preparation procedure, where $\mathcal{H}_{\rm prep}$ differs from the evolution Hamiltonian in a well-defined manner. In any case we shall verify that a Gaussian wave packet is indeed like taking a preparation with $\lambda \ll 1$.

In our numerical investigation we use the model Hamiltonian

$$\mathcal{H}(Q, P; x) = \tfrac{1}{2}(P_1^2 + P_2^2 + Q_1^2 + Q_2^2) + x Q_1^2 Q_2^2, \tag{8}$$

with $x = 1 + \delta x$. It describes the motion of a particle in a 2D well (2DW). The physical units are chosen so as to have dimensionless variables. Therefore upon quantization the Planck constant $\hbar$ is a dimensionless quantity. Our numerical study is focused on an energy window around $E \approx 3$ where the motion is mainly chaotic with characteristic correlation time $\tau_{\rm cl} \approx 1$ [14]. The quantization is done with $\hbar = 0.012$. We write the Hamiltonian matrix as in Eq. (7), using a basis such that $\mathcal{E}$ is diagonal. The mean level spacing is $\Delta \approx 4.3 \times \hbar^2$. As expected, on the basis of a general "quantum chaos" argumentation [15], the matrix $\mathcal{B}$ is a banded matrix. More details regarding the band profile can be found in Ref. [14]. The only additional piece of information that is needed for the following analysis is the parametric scale δx_c. This is defined as the δx which is needed in order to mix neighboring levels. It is given by the ratio Δ/σ, where σ is the root mean square value of the near diagonal matrix elements of the $\mathcal{B}$ matrix. For the above model $\delta x_c \approx 3.8\hbar^{3/2}$.

Figure 1 (left panels) displays representative results of simulated time reversal experiments. One experiment is done with a coherent state preparation, and we indeed see behavior that looks like an echo ($t_r \sim T$). Qualitatively the same behavior is observed for a random preparation that has $\lambda \ll 1$. Once we take a preparation with a larger value of λ, we realize that the compensation time is in general $t_r < T$. In particular with the $\lambda > 1$ preparation we do not observe any quantum reversibility ($t_r = T/2$).

In Fig. 2 we present some results for t_r/T. We clearly see that t_r is smaller for larger values of $\varepsilon_{\rm evol}$. Much more

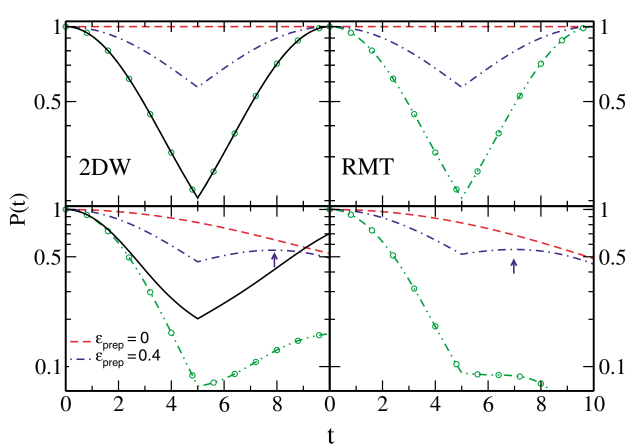

FIG. 1 (color online). Left panels: The probability $P(t)$ to find the system in its original state, for the 2DW model. The two panels, from top to bottom, are for the representative values $\varepsilon_{\rm evol} = 0.003\,15$ and 0.26. We use various preparations: a coherent state preparation (solid lines), a random superposition with the same energy width (○), a random superposition (dot-dashed lines) with a larger λ where $\epsilon_{\rm prep} = 0.4$, and a $\lambda > 1$ preparation (dashed lines) which was obtained by setting $\epsilon_{\rm prep} = 0$. The arrows indicate the time t_r for the case $\epsilon_{\rm prep} = 0.4$. Right panels: The simulations are now done with the corresponding RMT model.

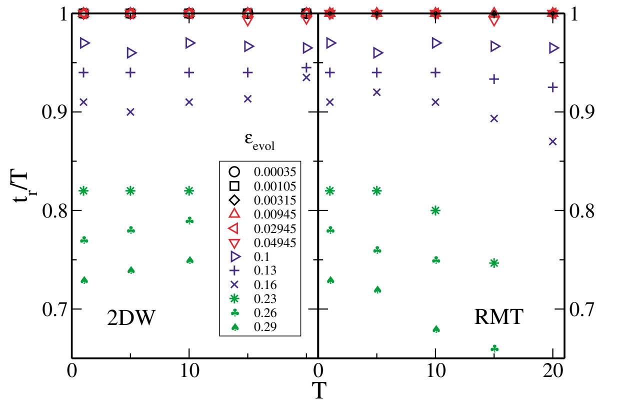

FIG. 2 (color online). The time t_r is calculated from the simulation of $P(t)$, and the ratio t_r/T is presented for various values of the period T, and for various values (see legend) of $\varepsilon_{\mathrm{evol}}$. An average over a number of initial wave packets with the same $\varepsilon_{\mathrm{prep}} = 0.4$ was performed. We clearly see that t_r is smaller for larger values of $\varepsilon_{\mathrm{evol}}$. The left panel corresponds to the 2DW simulations, while the right panel corresponds to the effective RMT model.

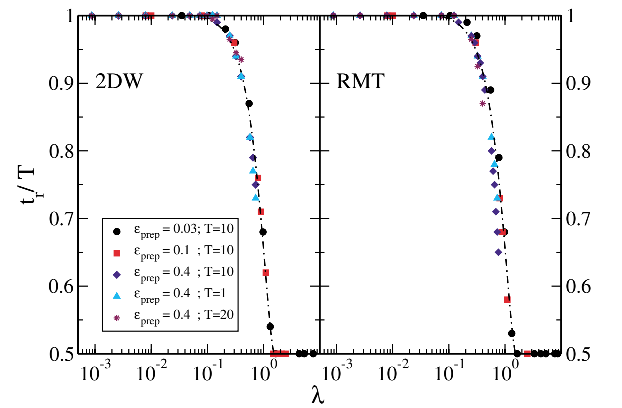

FIG. 3 (color online). The ratio t_r/T for different initial random wave packets which are determined by $\varepsilon_{\mathrm{prep}}$ and for various $\varepsilon_{\mathrm{evol}}$. The horizontal axis is the scaled variable λ. The data scale nicely in accordance with Eq. (6). The left panel corresponds to the 2DW simulations, while the right panel corresponds to the effective RMT model. The dot-dashed line (left panel) is a simple polynomial fit to the data. The same line is displayed in the right panel to allow comparison.

illuminating is Fig. 3 where we present several data sets for t_r. Each data set is characterized by a different $\varepsilon_{\mathrm{prep}}$ and T, while $\varepsilon_{\mathrm{evol}}$ is varied. The data points scale nicely with respect to λ, hence confirming the scaling hypothesis Eq. (6). Moreover, we see that for $\lambda > \lambda^*$, with $\lambda^* \approx 1.3$, we have $t_r = T/2$ (no echo) irrespective of the actual value of $\varepsilon_{\mathrm{prep}}$ and $\varepsilon_{\mathrm{evol}}$. Thus, if we want a $\lambda > \lambda^*$ type of preparation, we simply can set $\varepsilon_{\mathrm{prep}} = 0$, which means an eigenstate of $\mathcal{E}$. From Fig. 3 we also see that in the other limiting case of $\lambda \ll 1$ we have (nearly) an echo. This is the same as in the case of a Gaussian wave packet preparation.

In order to explain the observed results for t_r we look at Fig. 4. There are two axes along which the theory of $P(t_1, t_2)$ is quite well known. One is the traditional wave packet dynamics axis $t_2 = 0$ along which $P_{\mathrm{SR}}(t)$ is defined, while the other is the "LE axis" ($t_2 = t_1$) along which $P_{\mathrm{LE}}(t)$ is defined. In the figure we also indicate the course of a time reversal experiment. It is clear that in order to have $t_r < T$ the contour line $P(t_1, t_2) = P_{\mathrm{SR}}(T/2)$ should meet the t_1 axis in a sharp angle. Furthermore, if we want to have some remnant of an echo at the end of the period ($t = T$) we have to cross the contour line $P(t_1, t_2) = P_{\mathrm{LE}}(T/2)$. The condition for that is

$$P_{\mathrm{SR}}(T/2) < P_{\mathrm{LE}}(T/2). \tag{9}$$

In the following discussion we assume that both $\varepsilon_{\mathrm{prep}}$ and $\varepsilon_{\mathrm{evol}}$ are larger than δx_c, which means that the perturbations are strong enough to mix levels. In such case the decay of $P_{\mathrm{SR}}(t)$ or $P_{\mathrm{LE}}(t)$ is approximately exponential:

$$P_{\mathrm{SR}}(t), P_{\mathrm{LE}}(t) = \exp(-\gamma t). \tag{10}$$

Moreover in both cases γ is given by the expression

$$\gamma = \min(\gamma_{\mathrm{PT}}, \gamma_{\mathrm{SC}}), \tag{11}$$

where γ_{PT} is the value which is determined by perturbation theory, while γ_{SC} is determined by semiclassical considerations.

The detailed theories behind the exponential approximation Eq. (10) are quite different in the two respective cases [$P_{\mathrm{SR}}(t)$, $P_{\mathrm{LE}}(t)$]. The theory of the survival probability is related to the parametric theory of the local density of states (LDOS) [14,16]. Namely, for relatively small perturbations $\gamma = \gamma_{\mathrm{PT}} \propto (\varepsilon_{\mathrm{prep}}/\delta x_c)^2$ is essentially the width of Wigner's Lorentzian, while for large perturbations $\gamma = \gamma_{\mathrm{SC}} \propto \varepsilon_{\mathrm{prep}}$ is the width of the energy shell (in the latter case the exponential approximation is at best a good fit). In contrast to that, the theory of the fidelity $P_{\mathrm{LE}}(t)$ is related to a theory of dynamical correlations and cannot be reduced to the LDOS analysis [8]. The best theory to date is semiclassical (see [9] and references

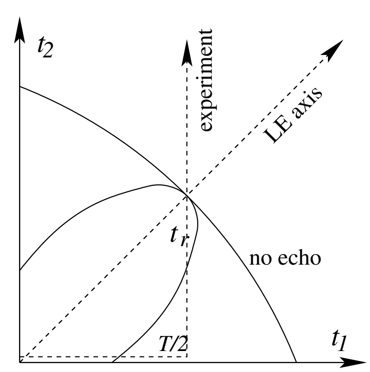

FIG. 4. The contour line of $P(t_1, t_2)$ that goes through the point $(T/2, T/2)$. One solid line ("no echo") is for the case $\lambda \geq 1$. The other solid line is for the case of a relatively small λ. The dashed line illustrates the course of a time reversal experiment, while the "LE axis" is the line along which P_{LE} is defined.

therein). As in the case of the survival probability we have $\gamma = \gamma_{\mathrm{PT}} \propto (\varepsilon_{\mathrm{evol}}/\delta x_c)^2$ for small perturbations, while for large perturbations $\gamma = \gamma_{\mathrm{SC}}$, in some typical cases, is related to the Lyapunov exponent. We point out that the study of the general conditions for having a fingerprint of the Lyapunov exponent in time reversal experiments is still an open issue for future study [17]. The existing semiclassical theory for the "echo" phenomena assumes Gaussian wave packets.

By definition $P_{\mathrm{SR}}(T/2)$ depends only on $\varepsilon_{\mathrm{prep}}$, while $P_{\mathrm{LE}}(T/2)$ is mainly sensitive to $\varepsilon_{\mathrm{evol}}$. Therefore it is evident that a small λ is a condition for having $t_r < T$. If we have $\lambda \ll 1$ then the contour line $P(t_1, t_2) = P_{\mathrm{LE}}(T/2)$ can be very close to the LE axis. This implies that for $\lambda \ll 1$ we can get nearly a perfect echo behavior ($t_r \sim T$). This picture, as we have seen before, is supported by our numerical findings.

As we have seen above, the applicability of semiclassical considerations is not essential for having an echo. The general picture that we have outlined should be valid also in the absence of a semiclassical limit. This is in contrast to the impression that one might get from the recent literature. In order to establish this provocative statement, in a way that leaves no doubts, we use a simple random matrix theory (RMT) procedure. We take the resulting banded matrix $\mathcal{B}$ of the 2DW model (8) and randomize the signs of the off-diagonal terms. In this way we get an effective RMT (ERMT) model of the type that was introduced by Wigner 50 years ago [18]. The model is characterized by the same mean level spacing and by the same band profile as the physical 2DW model. Consequently the generated dynamics is characterized by the same correlation time (the latter is determined by the bandwidth). But unlike the 2DW model, the ERMT model is lacking a semiclassical limit. In the right panels of Figs. 1–3 we demonstrate the results of simulations that were done with the ERMT model. We clearly see that we get similar results (in Fig. 3 the RMT drop is slightly sharper).

In summary, we have shed new light on the physics of quantum reversibility, and, in particular, we have introduced the concept of compensation time t_r, which replaces the misleading terminology of echo. Our predictions should be tested in wave field evolution experiments such as spin polarization echoes in nuclear magnetic resonances [10,19]. In particular, we have considered the realistic case of a general preparation and clarified the role of semiclassical considerations in the theory.

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