Renormalization of the dephasing by zero point fluctuations

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We study the role of zero-point-fluctuations (ZPF) in dephasing at low temperature. Unlike the Caldeira-Leggett model where the interaction is with an homogeneous fluctuating field of force, here we consider the effect of short range scattering by localized bath modes. We find that in presence of ZPF the inelastic cross-section gets renormalized. Thus indirectly ZPF might contribute to the dephasing at low temperature.

I. INTRODUCTION

The coherent motion of a quantum mechanical particle in a fluctuating environment is endangered by decoherence due to inelastic scattering events. The temperature dependence of the resulting "dephasing" effect has been worked out in numerous studies[1,2,3]. During the last decade a controversy has emerged in the mesoscopic literature regarding the role of zero-point-fluctuations (ZPF) in the theory of low temperature dephasing. The controversy was sparked by the experiment of Ref.[4] where a saturation of the dephasing rate in the limit of zero temperature has been reported, and consequently ZPF induced dephasing has been suggested as an explanation[5] and debated[6,7,8,9].

Possibly one can insist that ZPF lead to "$T = 0$" dephasing for a Brownian particle that interacts with an Ohmic Caldeira-Leggett (CL) bath[10] where the fluctuations of the environment consist of long wavelength ($q = 0$) modes. However[11,12], in metallic environment the effective fluctuations are characterized by a finite correlation distance, and hence consist of modes with wavenumbers q that range up to the Fermi momentum. It was largely accepted[7,9] that if the interactions are short range, such that the fluctuations are characterize by a finite correlation distance, then the effect of the ZPF would be to renormalize the scattering cross-section and the mass of the particle. It turns out that in the cases of physical interest, and in particular for the prototype model of Refs.[15,16], this renormalization effect is non-diverging in the zero temperature limit: both mass renormalization[17] and dephasing[18] for a single particle in the presence of a dirty metal environment have been studied. Essentially the same formulation as in Refs.[9,18] arise also in the more complicated many body treatment of the dephasing problem[13].

Though it is not diverging in problems of physical interest, still the renormalization effect due to ZPF might be significant in the actual analysis. The simplest possibility is to have an overall suppression of both elastic and inelastic scattering via a *Debye-Waller factor*[14]. But we would like to explore the more exotic possibility of having a distinct enhancement "factor" for the inelastic effect. It is therefore desired to have at least one exactly solvable model for dephasing due to short range scattering with environmental modes, that can be contrasted with the opposite CL limit where all the mode have $q = 0$. The objective of the present paper is to present such a model: In the proposed model (see Fig. 1) the environment consists of infinitely many localized fluctuating modes with (say) Ohmic spectral function, while the interaction is of short range and described by $\delta(x)$ as in "s-scattering". This should be contrasted with the long range interaction of the CL model which is linear in x.

The outline of this paper is as follows: In Section 2 we define a model for a localized bath that induces both *zero point fluctuations* (ZPF) and *thermal fluctuations* (TRF). In Sections 3-4 explicit expression for its scattering matrix are derived following[19]. In Section 5-6 low temperatures are considered, where the TRF are treated as a small perturbation. The example and the numerical analysis in Sections 7-8 establish that for weak TRF the effect of the ZPF background can be taken into account by defining a renormalized intensity of the TRF. Accordingly ZPF may contribute to the dephasing at low temperatures, though not directly.

II. THE MODEL

The Hamiltonian of the particle plus the local bath is

$$\mathcal{H} = \frac{p^2}{2\mathsf{m}} + \delta(x) \sum_\alpha c_\alpha Q_\alpha + \sum_\alpha \hat{n}_\alpha \omega_\alpha \tag{1}$$

The index $n_\alpha = 0, 1, 2, 3...$ may indicate the state of the α oscillator, or optionally $n_\alpha = 0, 1$ may indicate the state of a two level ("spin") entity, as in our numerics. From now on we use the notation

$$\hat{Q} = \sum_\alpha c_\alpha Q_\alpha \tag{2}$$

Assuming an incident particle with kinetic energy ϵ_k we divide the oscillators into two groups: those with $\omega_\alpha < \epsilon_k$ and those with $\omega_\alpha > \epsilon_k$ respectively. We further assume low temperatures such that all the oscillators in the latter group are in the ground state. Note that the particle has enough energy to induce real (non-virtual) excitation of any of the TRF oscillators. Hence we can write schematically:

$$\hat{Q} = \quad c_{\rm S} + \sum_{\alpha\in\mathrm{ZPF}} c_\alpha Q_\alpha + \sum_{\alpha\in\mathrm{TRF}} c_\alpha Q_\alpha \tag{3}$$

where $c_{\rm S}$ represents a static scatterer. The particle is affected by the fluctuations of Q. Assuming that the bath is prepared in the state $n = m$ the fluctuations are characterized by the non-symmetrized power spectrum

$$\tilde{S}(\omega) = \overline{\sum_{n(\neq m)} |Q_{nm}|^2 \, 2\pi\delta\left(\omega - (E_n - E_m)\right)} \quad (4)$$

In the next section we show how the ZPF oscillators can be eliminated, such that the interaction is characterized by a dressed interaction matrix $\mathcal{Q}$. Accordingly we define the effective power spectrum as

$$\tilde{S}_{\rm eff}(\omega) = \overline{\sum_{n(\neq m)} |\mathcal{Q}_{nm}|^2 \, 2\pi\delta\left(\omega - (E_n - E_m)\right)} \quad (5)$$

and the *effective* "size" of the elastic scatterer as

$$c_{\rm eff} = \mathcal{Q}_{m,m} \quad (6)$$

In the following sections we explain how to define $\mathcal{Q}$ and how to make the exact calculation of the elastic scattering amplitude $\mathcal{T}$, and of the inelastic scattering cross section $p_{\rm inelastic}$ (see Fig. 1). Then we discuss whether the results can be deduced from the effective values of $c_{\rm eff}$ and $\tilde{S}_{\rm eff}(\omega)$.

III. THE SCATTERING STATES

Outside of the scattering region the total energy of the system (particle plus bath) is

$$\mathcal{E} \;=\; \epsilon_k + E_n \;=\; \epsilon_k + \sum_\alpha n_\alpha \omega_\alpha \quad (7)$$

We look for scattering states that satisfy the equation

$$\mathcal{H}|\Psi\rangle = \mathcal{E}|\Psi\rangle$$

Open (propagating) channels are those for which $\epsilon_k > 0$ after scattering. Otherwise the channels are closed (evanescent). The channels are labeled as

$$\begin{aligned} \boldsymbol{n} &= (n_0, n) = (n_0, n_{\rm ZPF}, n_{\rm TRF}) \\ &= (n_0, n_1, n_2, n_3, ..., n_\alpha, ...) \end{aligned} \quad (8)$$

where $n_0 =$ L,R for left/right, and $n_{\rm ZPF}, n_{\rm TRF}$ are collective indexes for the two group of scatterers. We define

$$k_{\boldsymbol{n}} = \sqrt{2m(\mathcal{E} - E_n)} \qquad \text{for } n \in \text{open} \quad (9)$$
$$\alpha_{\boldsymbol{n}} = \sqrt{-2m(\mathcal{E} - E_n)} \qquad \text{for } n \in \text{closed} \quad (10)$$

later we use the notations

$$v_{\boldsymbol{n}} = k_{\boldsymbol{n}}/\mathsf{m} \quad (11)$$
$$u_{\boldsymbol{n}} = \alpha_{\boldsymbol{n}}/\mathsf{m} \quad (12)$$

and define diagonal matrices $\boldsymbol{v} = \text{diag}\{v_n\}$ and $\boldsymbol{u} = \text{diag}\{u_n\}$. The channel radial functions are written as

$$R(r) = A_n e^{-ik_n r} + B_n e^{+ik_n r} \qquad n \in \text{open} \quad (13)$$
$$R(r) = C_n e^{-\alpha_n r} \qquad n \in \text{closed} \quad (14)$$

where $r = |x|$. The wavefunction can be written as

$$\Psi(r, n_0, Q) = \sum_n R_{n_0,n}(r)\chi^n(Q) \quad (15)$$

The matching equations are

$$\Psi(0, \text{right}, Q) - \Psi(0, \text{left}, Q) \;=\; 0 \quad (16)$$
$$\frac{1}{2\mathsf{m}}\left[\Psi'(0, \text{right}, Q) + \Psi'(0, \text{left}, Q)\right] \;=\; \hat{Q}\Psi(0, Q) \quad (17)$$

The operator $\hat{Q}$ is represented by the matrix Q_{nm} that has the block structure

$$Q_{nm} = \begin{pmatrix} Q_{vv} & Q_{vu} \\ Q_{uv} & Q_{uu} \end{pmatrix} \quad (18)$$

The matching conditions lead to the following set of matrix equations

$$\begin{aligned} A_{\rm R} + B_{\rm R} &= A_{\rm L} + B_{\rm L} \\ C_{\rm R} &= C_{\rm L} \\ -i\boldsymbol{v}(A_{\rm R} - B_{\rm R} + A_{\rm L} - B_{\rm L}) &= 2Q_{vv}(A_{\rm L} + B_{\rm L}) + 2Q_{vu}C_{\rm L} \\ -\boldsymbol{u}(C_{\rm R} + C_{\rm L}) &= 2Q_{uv}(A_{\rm L} + B_{\rm L}) + 2Q_{uu}C_{\rm L} \end{aligned}$$

From here we get the matching equations that relate the ingoing and the outgoing amplitudes:

$$A_{\rm R} + B_{\rm R} \;=\; A_{\rm L} + B_{\rm L} \quad (19)$$
$$A_{\rm R} - B_{\rm R} + A_{\rm L} - B_{\rm L} \;=\; i2(\boldsymbol{v})^{-1}\mathcal{Q}(A_{\rm L} + B_{\rm L}) \quad (20)$$

where the dressed interaction matrix is defined as

$$\mathcal{Q} = Q_{vv} - Q_{vu}\frac{1}{(\boldsymbol{u} + Q_{uu})}Q_{uv} \quad (21)$$

In the next section we deduce the $\boldsymbol{S}$ matrix from the above set of equations, and obtain explicit expressions for the elastic scattering amplitude and for the inelastic cross section.

IV. THE S MATRIX

The unitary description of the scattering in terms of ingoing and out going probability currents requires to define the normalized ingoing and the outgoing amplitudes as $\tilde{A}_n = \sqrt{v_n}A_n$ and $\tilde{B}_n = \sqrt{v_n}B_n$. Consequently we defined a re-scaled version of the Q_{nm} matrix as follows:

$$M_{nm} = \begin{pmatrix} M_{vv} & M_{vu} \\ M_{uv} & M_{uu} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{v}}Q_{vv}\frac{1}{\sqrt{v}} & \frac{1}{\sqrt{v}}Q_{vu}\frac{1}{\sqrt{u}} \\ \frac{1}{\sqrt{u}}Q_{uv}\frac{1}{\sqrt{v}} & \frac{1}{\sqrt{u}}Q_{uu}\frac{1}{\sqrt{u}} \end{pmatrix} \quad (22)$$

We also define a corresponding reduced matrix

$$\mathcal{M} = \frac{1}{\sqrt{\boldsymbol{v}}}\mathcal{Q}\frac{1}{\sqrt{\boldsymbol{v}}} = M_{vv} - M_{vu}\frac{1}{1+M_{uu}}M_{uv} \tag{23}$$

Using these notations the set of matching conditions can be expressed using a transfer matrix as follows:

$$\begin{pmatrix}\tilde{B}_{\mathrm{R}}\\ \tilde{A}_{\mathrm{R}}\end{pmatrix} = \boldsymbol{T}\begin{pmatrix}\tilde{A}_{\mathrm{L}}\\ \tilde{B}_{\mathrm{L}}\end{pmatrix} \tag{24}$$

The transfer $2N \times 2N$ matrix can be written in block form as follows:

$$\boldsymbol{T} = \begin{pmatrix}\boldsymbol{T}_{++} & \boldsymbol{T}_{+-}\\ \boldsymbol{T}_{-+} & \boldsymbol{T}_{--}\end{pmatrix} = \begin{pmatrix}1-i\mathcal{M} & -i\mathcal{M}\\ i\mathcal{M} & 1+i\mathcal{M}\end{pmatrix} \tag{25}$$

The $\boldsymbol{S}$ matrix is defined via

$$\begin{pmatrix}\tilde{B}_{\mathrm{L}}\\ \tilde{B}_{\mathrm{R}}\end{pmatrix} = \boldsymbol{S}\begin{pmatrix}\tilde{A}_{\mathrm{L}}\\ \tilde{A}_{\mathrm{R}}\end{pmatrix} \tag{26}$$

and can be written in block form as

$$\boldsymbol{S}_{\boldsymbol{n},\boldsymbol{m}} \;=\; \begin{pmatrix}\boldsymbol{S}_R & \boldsymbol{S}_T\\ \boldsymbol{S}_T & \boldsymbol{S}_R\end{pmatrix} \tag{27}$$

A straightforward elimination gives

$$\boldsymbol{S} \;=\; \begin{pmatrix} -\boldsymbol{T}_{--}^{-1}\boldsymbol{T}_{-+} & \boldsymbol{T}_{--}^{-1}\\ \boldsymbol{T}_{++}-\boldsymbol{T}_{-+}\boldsymbol{T}_{--}^{-1}\boldsymbol{T}_{+-} & \boldsymbol{T}_{+-}\boldsymbol{T}_{--}^{-1}\end{pmatrix} \tag{28}$$

Now we can write expressions for $\boldsymbol{S}_R$ and for $\boldsymbol{S}_T$ using the $\mathcal{M}$ matrix.

$$\boldsymbol{S}_T \;=\; \frac{1}{1+i\mathcal{M}} = 1 - i\mathcal{M} - \mathcal{M}^2 + i\mathcal{M}^3 + ... \tag{29}$$

$$\boldsymbol{S}_R \;=\; \boldsymbol{S}_T - \boldsymbol{1} \tag{30}$$

The elastic forward scattering amplitude is

$$\mathcal{T} \;=\; [\boldsymbol{S}_T]_{\boldsymbol{m},\boldsymbol{m}} \;=\; \left[\frac{1}{1+i\mathcal{M}}\right]_{\boldsymbol{m},\boldsymbol{m}} \tag{31}$$

The total elastic scattering probability is

$$\begin{aligned} p_{\text{elastic}} &= |\mathcal{T}|^2 + |\mathcal{T}-1|^2 \\ &= 1 - 2\left[\Re(\mathcal{T}) - |\mathcal{T}|^2\right] \end{aligned} \tag{32}$$

We observe that the inelastic scattering is isotropic and its line shape (per direction) is

$$p(\omega) = \sum_{\boldsymbol{n}(\neq\boldsymbol{m})} |[\boldsymbol{S}_T]_{\boldsymbol{n}\,\boldsymbol{m}}|^2\, 2\pi\delta(\omega - (E_{\boldsymbol{n}} - E_{\boldsymbol{m}})) \tag{33}$$

with the measure $d\omega/(2\pi)$. The total inelastic cross section is obtained by integration

$$\begin{aligned} p_{\text{inelastic}} &= 2\int \frac{d\omega}{2\pi} p(\omega) \\ &= 2\sum_{\boldsymbol{n}(\neq\boldsymbol{m})} |[\boldsymbol{S}_T]_{\boldsymbol{n},\boldsymbol{m}}|^2 \\ &= 2\sum_{\boldsymbol{n}(\neq\boldsymbol{m})} \left|\left[\frac{1}{1+i\mathcal{M}}\right]_{\boldsymbol{n},\boldsymbol{m}}\right|^2 \end{aligned} \tag{34}$$

One can verify the $p_{\text{inelastic}}$ and p_{elastic} sum up to unity, which is essentially the "optical theorem".

V. PERTURBATION THEORY

Since the temperature is low we treat the small effect of the TRF in leading order. We write

$$\begin{aligned} \hat{Q} &= \hat{Q}^{\text{ZPF}} \otimes \boldsymbol{1}^{\text{TRF}} + \boldsymbol{1}^{\text{ZPF}} \otimes \hat{Q}^{\text{TRF}} \\ &\equiv \hat{Q}^0 + \delta\hat{Q} \end{aligned} \tag{35}$$

where $\hat{Q}^0$ is the sum over the ZPF coordinates including the static scatterer c_{S}, while $\delta\hat{Q}$ is the sum over the TRF coordinates. For the reduced $\mathcal{Q}$ matrix we get:

$$\begin{aligned} \mathcal{Q} &= Q^0_{vv} + \delta Q_{vv} \\ &\quad - Q^0_{vu}\left(\frac{1}{\boldsymbol{u}+Q^0_{uu}} - \frac{1}{\boldsymbol{u}+Q^0_{uu}}\delta Q_{uu}\frac{1}{\boldsymbol{u}+Q^0_{uu}}\right)Q^0_{uv} \end{aligned} \tag{36}$$

We assume that all the "important" open modes are well above the the evanescent threshold. This means that a single TRF transition is not enough to push the scattered particle into an evanescent mode. Accordingly δQ_{uv} and δQ_{vu} are not included. With the same spirit we further assume that the TRF transitions hardly affect the evanescent velocity, hence

$$u_{n_{\text{ZPF}},n_{\text{TRF}}} \approx u_{n_{\text{ZPF}},0} \tag{37}$$

When calculating the matrix element $\mathcal{Q}_{\boldsymbol{nm}}$ the second term constitutes a sum over sequences $Q^0_{n,\nu}...(\delta Q)_{\nu',\mu'}...Q^0_{\mu,m}$. In order to have a non zero term, the TRF-oscillators of the ν state should remain in the same state as in the n state, while one ZPF oscillator of the ν state has to be excited. Similar observation applies to the states of the oscillators of the μ state. The TRF transitions are induced by δQ during the evanescent motion of the particle. Accordingly we deduce that

$$\begin{aligned} \mathcal{Q} = &\left(\left[Q^{\text{ZPF}}_{vv} - Q^{\text{ZPF}}_{vu}\left(\frac{1}{\boldsymbol{u}+Q^{\text{ZPF}}_{uu}}\right)Q^{\text{ZPF}}_{uv}\right]_{m,m}\right)\boldsymbol{1}^{\text{TRF}} \\ &+ \left(1 + \left[Q^{\text{ZPF}}_{vu}\left(\frac{1}{\boldsymbol{u}+Q^{\text{ZPF}}_{uu}}\right)^2 Q^{\text{ZPF}}_{uv}\right]_{m,m}\right)Q^{\text{TRF}} \end{aligned}$$

which can be written schematically as follows

$$\mathcal{Q} \;=\; c_0\boldsymbol{1}^{\text{TRF}} \;+\; \lambda_0 Q^{\text{TRF}} \tag{38}$$

where the effective elastic scattering amplitude, and the scaling factor of the inelastic effect are

$$c_0 \;=\; c_{\text{S}} - \left[Q^{\text{ZPF}}_{vu}\left(\frac{1}{\boldsymbol{u}+Q^{\text{ZPF}}_{uu}}\right)Q^{\text{ZPF}}_{uv}\right]_{m,m} \tag{39}$$

$$\lambda_0 \;=\; 1 + \left[Q^{\text{ZPF}}_{vu}\left(\frac{1}{\boldsymbol{u}+Q^{\text{ZPF}}_{uu}}\right)^2 Q^{\text{ZPF}}_{uv}\right]_{m,m} \tag{40}$$

With an appropriate counter term we can make $c_0 = 0$. More interestingly we see that the effective TRF are characterized by the dressed power spectrum

$$\tilde{S}_{\text{eff}}(\omega) \;=\; (\lambda_0)^2\,\tilde{S}(\omega) \tag{41}$$

VI. THE DRESSED BORN APPROXIMATION

The first order ("Born") approximation relates the inelastic line shape to the power spectrum of the fluctuations. We use the term "dressed Born approximation" in order to indicate that we use first order perturbation theory with respect to the TRF, while the ZPF including the static scatterer are treated to infinite order. Within this framework the leading order expression for the $\boldsymbol{S}$ matrix, using Eqs.(38-40), is

$$\begin{aligned}\boldsymbol{S}_T &\approx \frac{1}{1+i(c_0/v_\epsilon)+i\lambda_0\mathcal{M}^{\text{TRF}}} \\ &= \mathcal{T}_0\mathbf{1} - i\mathcal{T}_0^2\lambda_0\mathcal{M}^{\text{TRF}} + ... \end{aligned} \tag{42}$$

where the elastic forward scattering amplitude is

$$\mathcal{T}_0 = \frac{1}{1+i(c_0/v_\epsilon)} \tag{43}$$

Consequently we get for the inelastic scattering

$$p(\omega) \approx \frac{1}{v_\epsilon v_{\epsilon-\omega}}\,|\mathcal{T}_0|^4\,(\lambda_0)^2\,\tilde{S}(\omega) \tag{44}$$

Since we had assumed that the change in the kinetic energy of the particle due to TRF inelastic scattering is relatively small, one can take $v_{\epsilon-\omega} \approx v_\epsilon$.

VII. THE SIMPLEST EXAMPLE

Consider a particle with velocity v_ϵ, that collides with a 'bath' that consists of an elastic scatterer c_{S}, and a single two level TRF scatterer c_{T} whose excitation energy is $\omega_{\text{T}}(\ll \epsilon_k)$. The interaction matrix is

$$\mathcal{Q} = Q = \begin{pmatrix} c_{\text{S}} & c_{\text{T}} \\ c_{\text{T}} & c_{\text{S}} \end{pmatrix} \tag{45}$$

which we substitute in $\mathcal{M} \approx (1/v_\epsilon)^2\mathcal{Q}$, so as to get $\boldsymbol{S}_T = (1+i\mathcal{M})^{-1}$. In order to avoid crowded expressions we set the units such that $v_\epsilon = 1$, and write

$$\boldsymbol{S}_T = \frac{1}{(1+ic_{\text{S}})^2+c_{\text{T}}^2}\begin{pmatrix} 1+ic_{\text{S}} & -ic_{\text{T}} \\ -ic_{\text{T}} & 1+ic_{\text{S}} \end{pmatrix} \tag{46}$$

[Note again that in order to restore the units each c in the above expression should be replaced by c/v_ϵ]. From here it follows that

$$p_{\text{inelastic}} = \frac{2\nu_{\text{TRF}}}{(1-c_0^2+\nu_{\text{TRF}})^2+4c_0^2} \qquad \text{[scaled]} \tag{47}$$

where $\nu_{\text{TRF}} \equiv c_{\text{T}}^2$ characterizes the intensity of the TRF, and $c_0 \equiv c_{\text{S}}$. One observes that for strong TRF the inelastic effect is suppressed and we get mainly elastic back reflection. But in the regime of interest, of weak TRF, the inelastic scattering is proportional to ν_{TRF} and agree with Eq.(44) where $|\mathcal{T}_0|^2 = 1/(1+c_0^2)$ and $\lambda_0 = 1$.

Next we complicate the 'bath' by adding a single ZPF scatterer c_{Z} whose excitation energy is $\omega_{\text{Z}}(>\epsilon_k)$. The possible values of the mode index are $n=(0,0)\equiv 1$, and $n=(0,1)\equiv 2$, and $n=(1,0)\equiv 3$, and $n=(1,1)\equiv 4$. The ZPF scatterer is assumed to be in the ground state ($m=1$), and hence only the first two modes are open. The interaction matrix is

$$Q = \begin{pmatrix} c_{\text{S}} & c_{\text{T}} & c_{\text{Z}} & 0 \\ c_{\text{T}} & c_{\text{S}} & 0 & c_{\text{Z}} \\ c_{\text{Z}} & 0 & c_{\text{S}} & c_{\text{T}} \\ 0 & c_{\text{Z}} & c_{\text{T}} & c_{\text{S}} \end{pmatrix}.$$

If we did not have the TRF oscillator, it would be a 2×2 matrix:

$$Q^{\text{ZPF}} = \begin{pmatrix} c_{\text{S}} & c_{\text{Z}} \\ c_{\text{Z}} & c_{\text{S}} \end{pmatrix} \tag{48}$$

If we ignored the ZPF oscillator, we would get Eq.(45). But using Eq.(21) we get the dressed interaction matrix:

$$\begin{aligned}\mathcal{Q} &= \begin{pmatrix} c_{\text{S}} & c_{\text{T}} \\ c_{\text{T}} & c_{\text{S}} \end{pmatrix} \\ &- \frac{c_{\text{Z}}^2}{(u_3+c_{\text{S}})\,(u_4+c_{\text{S}}) - c_{\text{T}}^2}\begin{pmatrix} (u_3+c_{\text{S}}) & -c_{\text{T}} \\ -c_{\text{T}} & (u_4+c_{\text{S}}) \end{pmatrix}\end{aligned}$$

with $u_3 = \sqrt{|\epsilon_k-\omega_{\text{Z}}|}$ and $u_4 = \sqrt{|\epsilon_k-\omega_{\text{Z}}-\omega_{\text{T}}|}$. Consequently from $c_{\text{eff}} \equiv \mathcal{Q}_{1,1}$ we get:

$$c_{\text{eff}} = c_{\text{S}} - \frac{(u_3+c_{\text{S}})\,\nu_{\text{ZPF}}}{(u_3+c_{\text{S}})\,(u_4+c_{\text{S}}) - \nu_{\text{TRF}}} \tag{49}$$

and from $\nu_{\text{eff}} \equiv |\mathcal{Q}_{2,1}|^2$ we get $\nu_{\text{eff}} = \lambda^2\nu_{\text{TRF}}$ where

$$\lambda = 1 + \frac{\nu_{\text{ZPF}}}{(u_3+c_{\text{S}})\,(u_4+c_{\text{S}}) - \nu_{\text{TRF}}} \tag{50}$$

with $\nu_{\text{ZPF}} \equiv c_{\text{Z}}^2$ and $\nu_{\text{TRF}} \equiv c_{\text{T}}^2$. Optionally we can get for $\mathcal{Q}$ the approximated result of Eq.(38), which treats the TRF coupling in leading order. This treatment assumes that in the vicinity of the energy shell $v_1 \approx v_2 \equiv v_\epsilon$, and $u_3 \approx u_4 \equiv u_{\text{Z}}$. The parameters c_0 and λ_0 are calculated using Eqs.(39-40) with $Q^{\text{ZPF}}_{uu} = c_{\text{S}}$, and $Q^{\text{ZPF}}_{vu} = Q^{\text{ZPF}}_{uv} = c_{\text{Z}}$, leading to

$$c_0 = c_{\text{S}} - \frac{\nu_{\text{ZPF}}}{(u_{\text{Z}}+c_{\text{S}})} \tag{51}$$

$$\lambda_0 = 1 + \frac{\nu_{\text{ZPF}}}{(u_{\text{Z}}+c_{\text{S}})^2} \tag{52}$$

In this simple example the dependence of c_0 and λ_0 on ν_{ZPF} is linear. But once we have more than one ZPF scatterer (as in the numerical example of the next section) the relation is no longer linear. It might be also of interest to solve the first equation $c_0 = 0$ for c_{S}, and substitute the result into the second equation. The outcome of this procedure is illustrated in Fig. 3.

The calculation of the $\boldsymbol{S}$ matrix proceed in the same way as in the single TRF case, with the effective interaction matrix (no approximation involved):

$$\mathcal{Q} = \begin{pmatrix} c_{\text{S}} - (\lambda-1)(u_3+c_{\text{S}}) & \lambda c_{\text{T}} \\ \lambda c_{\text{T}} & c_{\text{S}} - (\lambda-1)(u_4+c_{\text{S}}) \end{pmatrix} \tag{53}$$

Setting $v_1 \approx v_2 \equiv v_\epsilon$ and $u_3 \approx u_4 \equiv u_{\rm Z}$ as before, we label both diagonal terms as $c_{\rm eff}$. Still we are not making any approximation with regard to the intensities $\nu_{\rm TRF}$ and $\nu_{\rm ZPF}$, so as to get essentially exact results:

$$|\mathcal{T}|^2 \;=\; \frac{[v_\epsilon^2 + (c_{\rm eff})^2]\, v_\epsilon^2}{[v_\epsilon^2 - (c_{\rm eff})^2 + \lambda^2 \nu_{\rm TRF}]^2 + 4v_\epsilon^2 (c_{\rm eff})^2} \quad (54)$$

and the generalization of Eq.(47):

$$p_{\rm inelastic} \;=\; \frac{2}{v_\epsilon^2}\; |\tilde{\mathcal{T}}|^2 |\mathcal{T}|^2 \;\lambda^2 \;\nu_{\rm TRF} \quad (55)$$

where $|\tilde{\mathcal{T}}|^2$ is Eq.(54) without the $\lambda^2\nu_{\rm TRF}$ term. For weak TRF intensity, using $\lambda \approx \lambda_0$ and $\mathcal{T} \approx \mathcal{T}_0$ one obtains the dressed Born approximation Eq.(44). One observes that the presence of the factor λ has two implications: one is to enhance the inelastic scattering for weak TRF, while the other is to limit the range over which the weak TRF approximation applies.

VIII. DISCUSSION, EXPECTATIONS, AND NUMERICAL DEMONSTRATION

The analysis in the present paper is focused primarily on the low temperature scattering, due to weak TRF, where the dressed Born approximation of Section 6 applies. Still in order to get the "big picture" we consider below the full range of $\nu_{\rm TRF}$ values. We first highlight some qualitative observations that are based on the analysis of the simple examples of the previous section, and then proceed with a numerical demonstration that involves a larger bath of scatterers.

From the Born approximation we deduce that for weak TRF the inelastic cross section $p_{\rm inelastic}$ is proportional to $\nu_{\rm TRF}$. For strong TRF it drops down as implied e.g. by the simplest example Eq.(47). The maximum $p_{\rm inelastic} = 1/2$ is attained for the intermediate value $\nu_{\rm TRF} = c_0^2{+}1$. A-priory we could not expect a larger inelastic effect because the elastic cross-section $|\mathcal{T}|^2 + |\mathcal{T}{-}1|^2$ is bounded from below by the minimum value 50%. We can interpret the condition for attaining minimum elastic cross section using a *Fabry-Perrot* double barrier picture: The elastic scattering and the inelastic scattering are like two barriers separated by an infinitesimal distance. The strongest interference effect is expected when the two barriers are comparable.

The suppression of the inelastic effect for strong TRF is a generic effect: it becomes almost obvious if we consider the scattering of a particle from a fluctuating region in a three dimensional space. In the latter context strong fluctuations would repel the particle from the scattering region, hence making inelastic excitations within the excluded volume less likely. So the strongest inelastic effect is experienced for intermediate values of $\nu_{\rm TRF}$.

The inclusion of ZPF into the model renormalizes $\nu_{\rm TRF}$. The enhancement factor λ is larger than unity (but finite) in the Born approximation limit, but if we go to very high temperatures (large $\nu_{\rm TRF}$) this renormalization effect fades away and we get $\lambda = 1$. See e.g. Eq.(50). The crossover involves a wild variation of λ (see Fig. 4), which implies that $0 < p_{\rm inelastic} < 1/2$ goes through the whole range of possible values (Fig. 5).

It is important to point out that if the fluctuations had continuous (rather than discrete) power spectrum, the above described intermediate wild variation would be smoothed away. Thus in realistic circumstances we expect that also in the presence of ZPF the qualitative dependence of $p_{\rm inelastic}$ on $\nu_{\rm TRF}$ would be smooth, though renormalized by λ_0 at the limit of low temperatures.

For the numerical study we consider a bath that consists of two level scatterers. The energy splitting of the α scatterer is ω_α and the interaction is described by the operator $Q_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The strength of the interaction with the bath is characterized by the intensity of the fluctuations as obtained by integrating over their power-spectrum $\tilde{S}(\omega)$. Consequently we distinguish between the intensity of the ZPF and the intensity of the TRF:

$$\nu_{\rm ZPF} \;=\; \sum_{\alpha \in ZPF} c_\alpha^2 \quad (56)$$

$$\nu_{\rm TRF} \;=\; \sum_{\alpha \in TRF} c_\alpha^2 \quad (57)$$

The *effective* intensity of the thermal fluctuations is similarly defined and accordingly calculated from the dressed interaction matrix:

$$\nu_{\rm eff} \;=\; \sum_{n (\neq m)} \left| \mathcal{Q}_{n,m} \right|^2 \quad (58)$$

Given a set of N ZPF-scatterers with couplings $c_{\rm Z}$, and a static scatterer $c_{\rm S}$, we calculate c_0 (which determines $\mathcal{T}_0$) and λ_0 as a function of $\nu_{\rm ZPF} \equiv N|c_{\rm Z}|^2$. See Fig. 6. Then, for various values of $\nu_{\rm ZPF}$ we calculate the exact results for $\nu_{\rm eff}$ and $p_{\rm inelastic}$ versus $\nu_{\rm TRF}$. See Fig. 7. One expects that for weak TRF the effective intensity $\nu_{\rm eff}$ would be proportional to $\nu_{\rm TRF}$, namely

$$\nu_{\rm eff} \;\approx\; (\lambda_0)^2 \;\nu_{\rm TRF} \quad (59)$$

Furthermore our perturbative scheme implies that

$$p_{\rm inelastic} \;\approx\; \frac{2}{v_\epsilon^2}\; |\mathcal{T}|^4 \;\nu_{\rm eff} \quad (60)$$

In order to test the quality of the latter approximation we re-plot the results for $p_{\rm inelastic}$ versus $|\mathcal{T}|^4 \nu_{\rm eff}$. See Fig. 8. The numerical results confirm our qualitative expectations, and are in agreements with the analysis of the simple example of the previous section. In particular one observes that the presence of ZPF has two implications: one is to enhance the inelastic scattering for weak TRF, while the other is to limit the range over which the weak TRF approximation applies.

IX. SUMMARY

In the Caldeira-Leggett model the effect of the environment is characterized by a friction coefficient η and by a temperature T. But more generally[9,11,12,18] it has been emphasized that the proper way to characterize the environment is by its form factor $\tilde{S}(q,\omega)$. The form factor contains information on both the temporal and the spatial aspects of the fluctuations, and in particular one can extract from it not only T and η, but also the spatial correlations. The general formula for the rate of dephasing[9,18] involves a $dqd\omega$ integral over $\tilde{S}(q,\omega)$, and for short range interactions simply reflect the rate of inelastic events.

We find for our model system that the inelastic scattering cross-section $p_{\text{inelastic}}$ is enhanced in the presence of ZPF, and accordingly ZPF may contribute to the dephasing at low temperatures, though indirectly. This might come as a surprise since in Ref.[14] it has been argued that both elastic and inelastic scattering are suppressed by ZPF by the *same* Debye-Waller factor (DWF). A closer look reveals the difference between the two models involved. In Ref.[14] one considers the scattering of a particle ($\hat{x}$) from a vibrating scatterer ($\hat{Q}$), where the interaction is $\delta(\hat{x}-\hat{Q})$. Accordingly the particle experiences a fluctuating field $\hat{\mathcal{U}}(x) = \delta(\hat{Q}-x)$ and $\tilde{S}(q,\omega)$ is the Fourier transform of $\langle \mathrm{e}^{-iq\hat{Q}(t)}\mathrm{e}^{iq\hat{Q}(0)}\rangle$, which is suppressed by the DWF $\mathrm{e}^{-\langle \hat{Q}^2 q^2\rangle}$. In our model the interaction is $\delta(\hat{x})\hat{Q}$. Accordingly $\tilde{S}(q,\omega)$ is the Fourier transform of $\langle \hat{Q}(t)\hat{Q}(0)\rangle$, and, within the framework of the conventional Born approximation, there is no DWF involved: in our model adding high frequency components to the fluctuating field has no implication on the low frequency behavior of $\tilde{S}(q,\omega)$.

The renormalization factor of the inelastic effect (λ) in our *dressed* Born approximation comes from higher orders of perturbation theory with respect to the ZPF, while the TRF are treated in leading order. The renormalization factor λ multiplies the power spectrum $\tilde{S}(\omega)$ that describes the thermal fluctuations. The power spectrum itself does not involve a DWF. The λ renormalization of the inelastic scattering comes "on top" of the expected renormalization of the *potential floor* and of the *inertial mass*, which are familiar from the solution of the Polaron problem. In our scattering theory framework the expected renormalization of the potential floor can be deduced from the ZPF induced offset in the effective 'size' of the elastic scatterer (c_{S}), while the renormalization of the mass comes from the associated energy dependence of the forward scattering amplitude ($\mathcal{T}$).

One can construct an extended bath that consists of an homogeneously distributed set of "s-scatterers", as described in[11]. This would allow the modeling of a fluctuating environment of physical interest (say a Dirty metal environment) with the desired $\tilde{S}(q,\omega)$. In such physical circumstances we expect renormalization of **(i)** the potential floor; **(ii)** the inertial mass; and **(iii)** the effective thermal fluctuations. Our results imply that these renormalization effects are non-divergent if the fluctuations are characterized by short range spatial correlations, but still they might modify the low temperature dependence of the dephasing effect.

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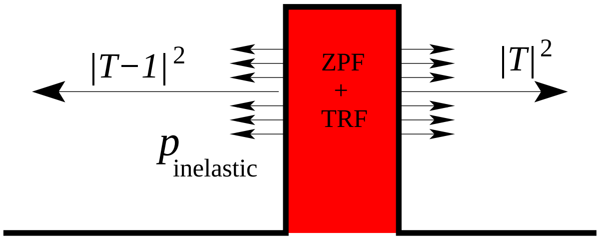

Fig.1: Schematic diagram of the model system. The scattered wave of a particle that collides from the right with a thermal "s-scatterer" consists of forward elastic scattering with amplitude $\mathcal{T}$, backward elastic scattering with amplitude $\mathcal{T}-1$, and isotropic inelastic scattering with probability $p_{\text{inelastic}}$. Our purpose is to find the dependence of $\mathcal{T}$ and $p_{\text{inelastic}}$ on the intensity of the low temperature thermal fluctuations (TRF), with arbitrarily large background of zero point fluctuations (ZPF).

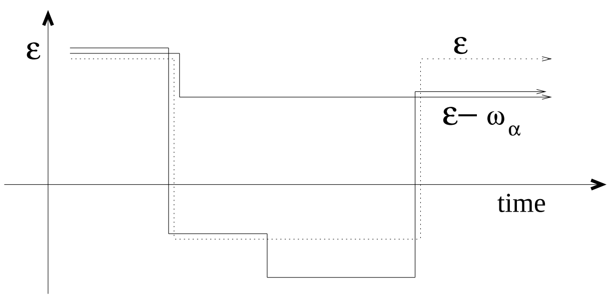

Fig.2: Diagrams that describe the time evolution of kinetic energy can be used in order to illustrate terms the scattering calculation. The dotted line represents a contribution to the elastic cross section due to (virtual) scattering by ZPF modes. The solid lines represent contributions to the first order inelastic cross-section, where the intensity of the TRF is regarded as the small parameter.

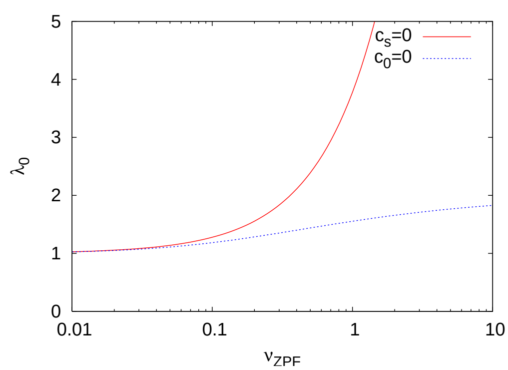

Fig.3: The renormalization factor for inelastic effect λ_{ZPF} is plotted as a function of ν_{ZPF} for the simple model of Section 7, using Eq.(52) with fixed static scatterer $c_{\text{S}}=0$ (solid red curve) and with adaptive static scatterer such that $c_0=0$ (dashed blue curve). The other parameters for this and for the next figures are $\omega_{\text{Z}}=0.96$ and $\omega_{\text{T}}=0.03$ and $\epsilon_k=0.6$.

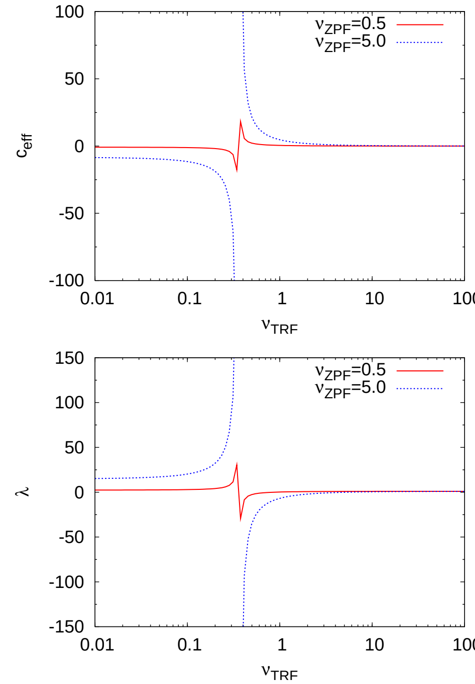

Fig.4: Plots of c_{eff} (upper panel) and λ (lower panel) versus ν_{TRF} for the simple model of Section 7 with the same parameters as in Fig. 3. The red solid curves are for $\nu_{\text{ZPF}}=0.5$ and the blue dashed curves are for $\nu_{\text{ZPF}}=5$.

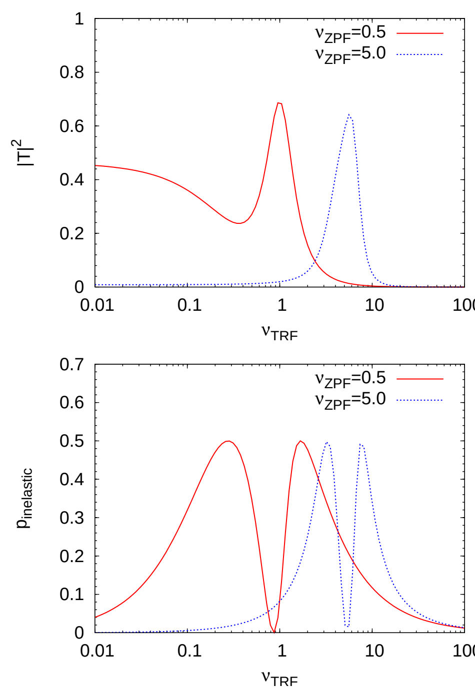

Fig.5: The transmission $|\mathcal{T}|^2$ (upper panel) and the inelastic cross section $p_{\text{inelastic}}$ (lower panel) versus ν_{TRF} for the simple model of Section 7 with the same parameters as in Figs. 3-4.

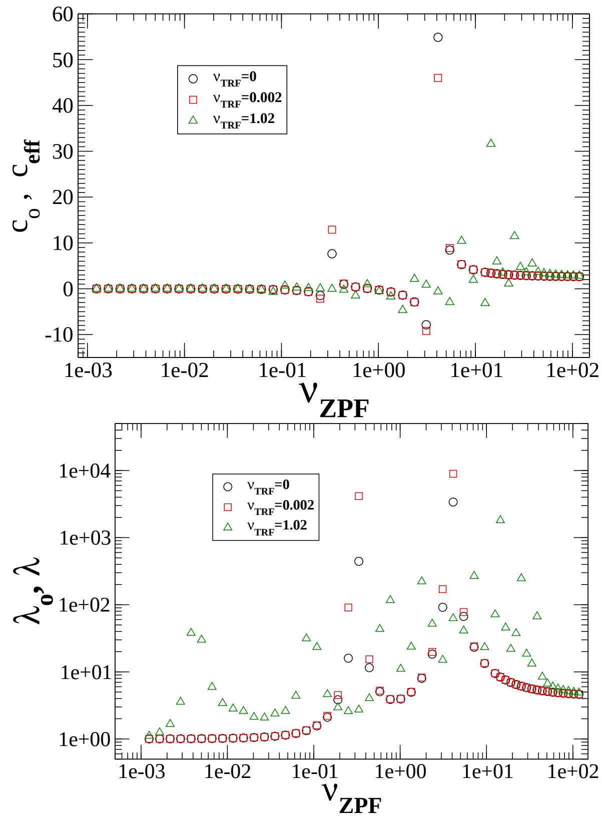

Fig.6: Plots of c_0 and λ_0 versus $\nu_{\rm ZPF}$. For sake of comparison we also plot $c_{\rm eff}$ and $\lambda \equiv (\nu_{\rm eff}/\nu_{\rm TRF})^{1/2}$ for two non-zero values of $\nu_{\rm TRF}$. Here and in the next figure we consider a bath that consists of 7 TRF scatterers with $\omega_\alpha \sim 0.0003$, and 4 ZPF scatterers with $\omega_\alpha \sim 0.96$. There is no static scatterer ($c_{\rm S} = 0$). The kinetic energy of the incident particle is $\epsilon_k = 0.6$.

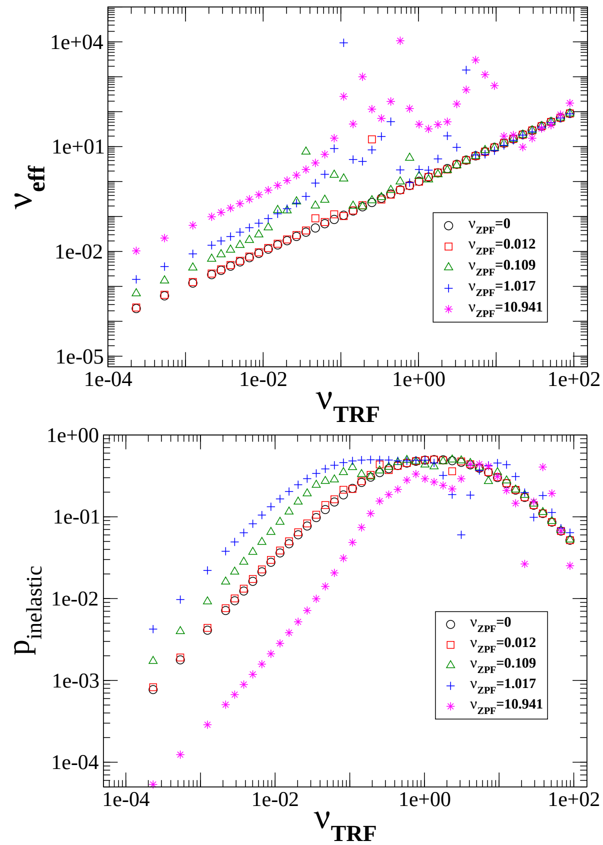

Fig.7: Plots of $\nu_{\rm eff}$ and $p_{\rm inelastic}$ versus $\nu_{\rm TRF}$ for the same bath as in the previous figure.

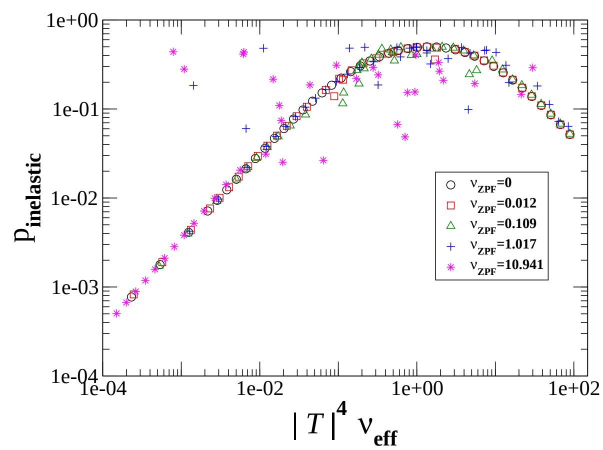

Fig.8: The inelastic cross section $p_{\rm inelastic}$ versus the scaled intensity of the thermal fluctuations $|\mathcal{T}|^4\nu_{\rm eff}$, using the data points of Fig. 7.