Celebrating 65th birthday of Rick Heller

The rate of heating in vibrating billiards

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Alex Barnett (Harvard 2000-2001)
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Tsampikos Kottos (Wesleyan)
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Michael Wilkinson (UK)
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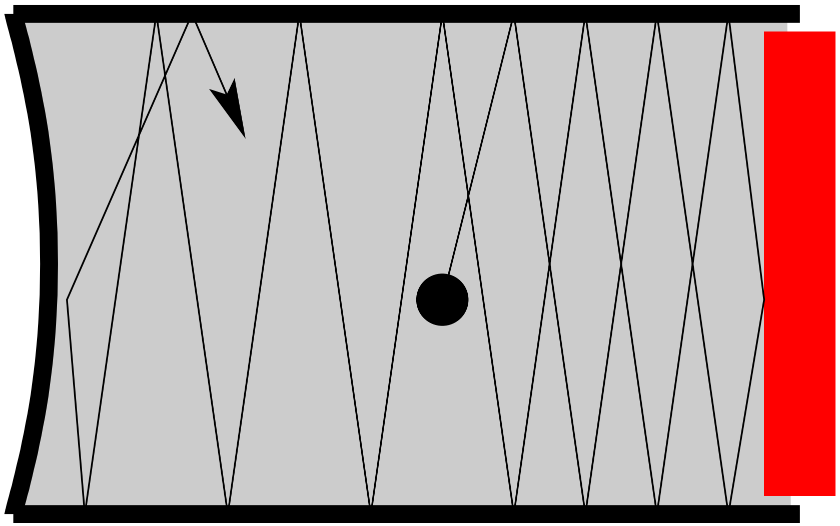
$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$



\$ISF, \$GIF, \$DIP, \$BSF

Dynamics and spectral intensities

$$\mathrm{FT}\Big[\langle\psi(0)|\psi(t)\rangle\Big] \quad\sim\quad \Big|\langle E_n|\psi\rangle\Big|^2$$

Semiclassical approaches to chaos 613

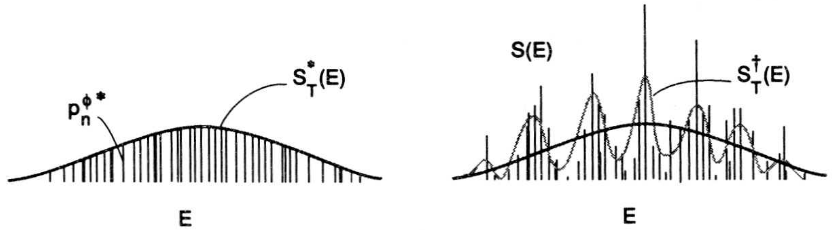


Fig. 38. Ideally ergodic (left) and typically found (right) spectral intensities and envelopes. Both spectra have the same low resolution envelope.

Les Houches 1989

Analogous relation between correlation function and band-profile:

$$\mathrm{FT}\Big[\langle V(0)V(t)\rangle\Big] \quad\sim\quad \Big|\langle E_n|\hat{V}|E_m\rangle\Big|^2$$

[Feingold-Peres]

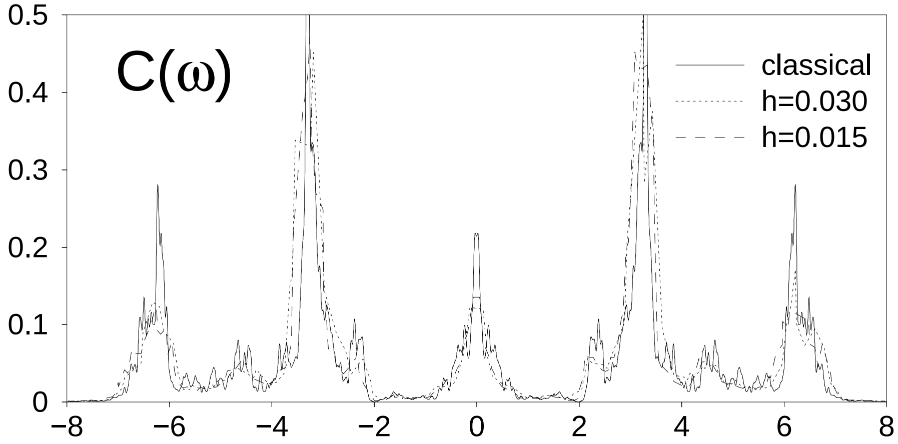
$$\tilde{C}(\omega) = \overline{\sum_n |V_{nm}|^2 \; 2\pi\delta(\omega - (E_n - E_m))}$$

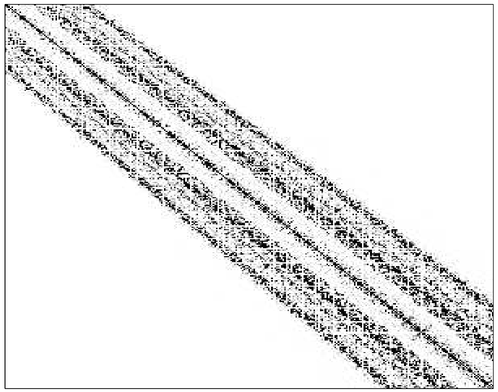
$$\rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \rightsquigarrow \qquad |V_{nm}|^2 \approx (2\pi\varrho)^{-1}\tilde{C}_{cl}(E_n - E_m)$$

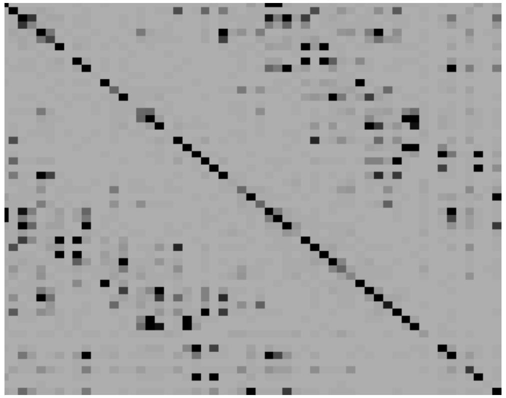
Bandprofile, sparsity and texture

$$|V_{nm}|^2 \approx (2\pi\varrho)^{-1}\tilde{C}_{cl}(E_n - E_m)$$

Hard Qchaos

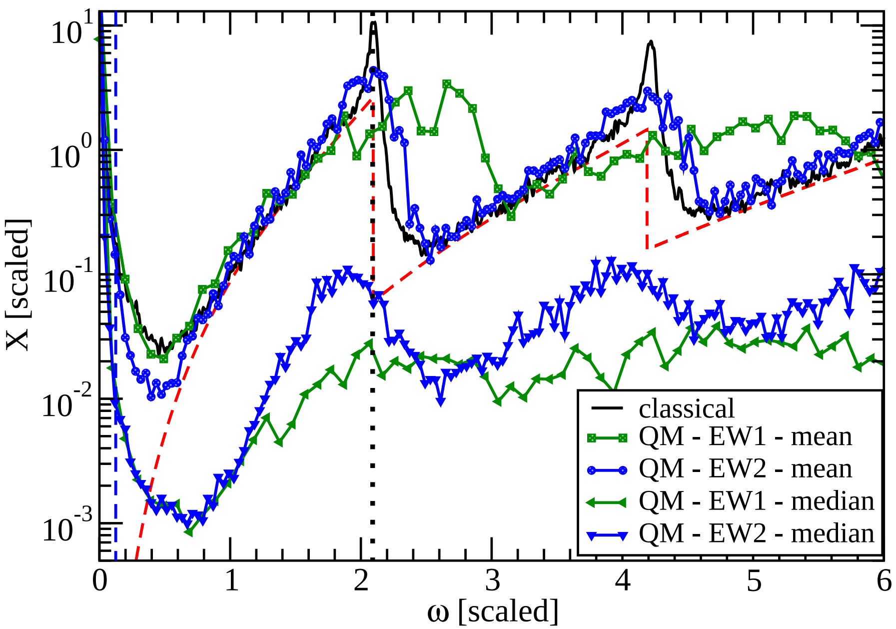


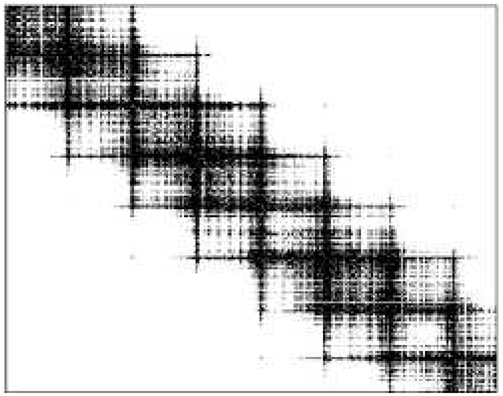




Weak Qchaos

[median ≪ mean]

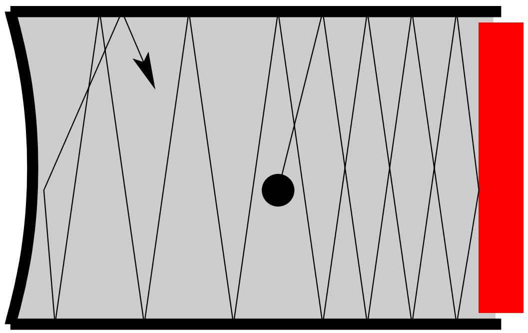




The rate of heating: LRT and SLRT predictions

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$

$f(t)$ = low freq noisy driving



$\rightsquigarrow$ diffusion in energy space:

$$D_0 = \frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\,\overline{\dot{f}^2}$$

$\rightsquigarrow$ energy absorption:

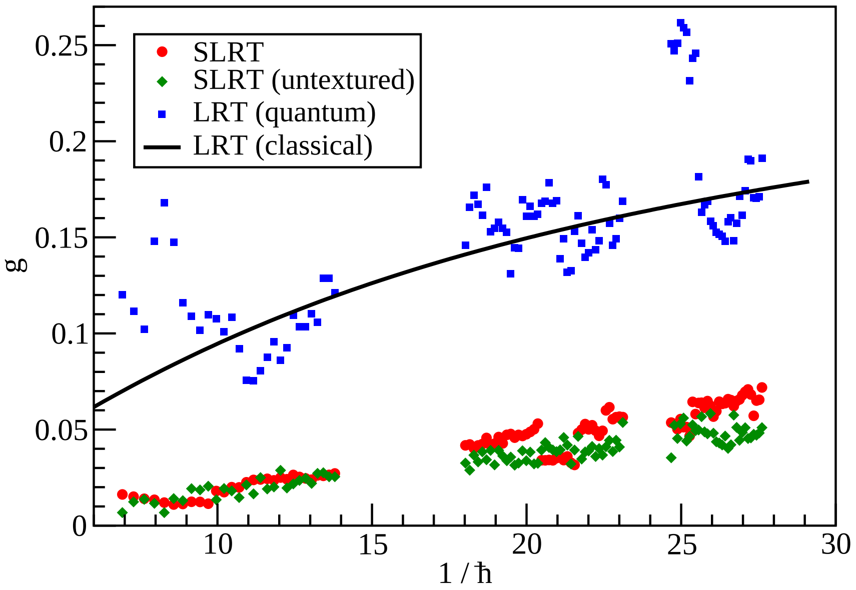
$$\dot{E} = (\text{particles/energy}) \times D$$

Beyond the "Wall Formula"

[Beyond the "Drude Formula"]

$$D_{\text{LRT}} = g_c\, D_0 \qquad [\text{"classical"}]$$

$$D_{\text{SLRT}} = g_s\, D_{\text{LRT}} \qquad [\text{"quantum"}]$$



LRT applies if the driven transitions are slower than the environmental relaxation, else SLRT applies

Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding g_c

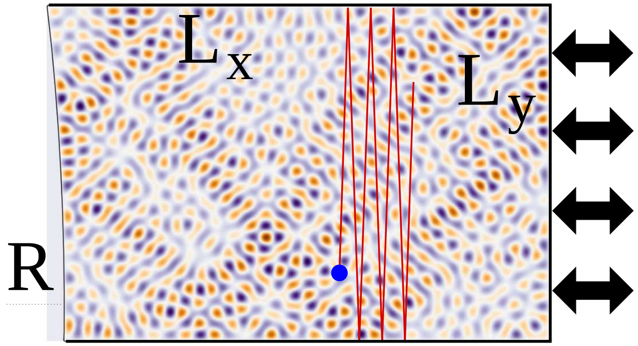
Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding g_s

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

$$u \;=\; (t_{\mathrm{R}}\,/\,t_{\mathrm{L}})^{-1} \;=\; (R/L)^{-1} \;=\; \text{deformation}$$

$$\hbar \;=\; \lambda_{\mathrm{E}}\,/\,L \;=\; 2\pi/(k_{\mathrm{E}}L) \;=\; \text{function of } E$$



[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)

[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)

[3] S. Bandopadhyay, Y. Etzioni, D. Cohen (EPL 2006)

[4] M. Wilkinson, B. Mehlig, D. Cohen (EPL 2006)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos, D. Cohen (JPA/FTC 2008)

[6] A. Stotland, T. Kottos, D. Cohen (PRB 2010)

[7] A. Stotland, D. Cohen, N. Davidson (EPL 2009)

[8] A. Stotland, L.M. Pecora, D. Cohen (arXiv 2010)

Digression - Bilha Segev (1963-2005)

Technion(1988-1996), Harvard (1996-1998), BGU (1998-2005).

Heating of particles by “shaking” the box

$$\mathcal{H}_{\text{total}} \approx \mathcal{H} + f(t)V$$

$f(t)$ = low freq noisy driving

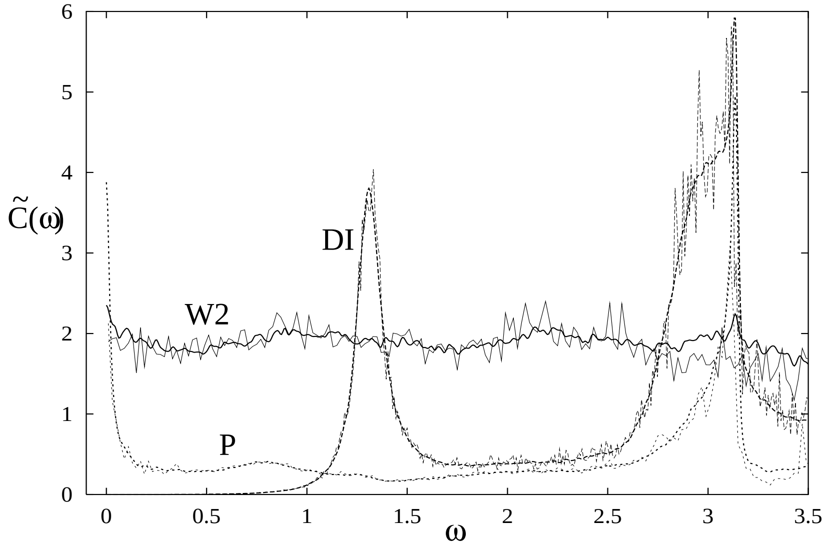
$$\tilde{C}(\omega) \;=\; \text{FT}\;\langle V(t)V(0)\rangle$$

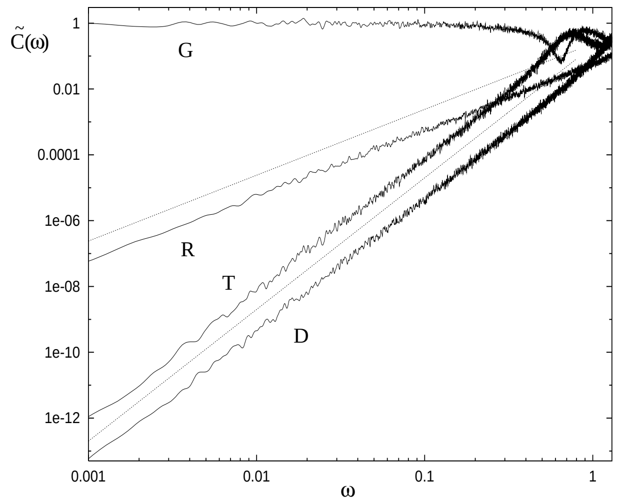
$$\tilde{S}(\omega) \;=\; \text{FT}\;\langle \dot{f}(t)\dot{f}(0)\rangle$$

Kubo formula:

$$\boldsymbol{D} = \int_0^\infty \tilde{C}(\omega)\tilde{S}(\omega)d\omega \;=\; g_c\,\frac{4}{3\pi}\,\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\,\overline{\dot{f}^2}$$

- $g_c \sim 1$ for “wiggle” deformation.
- $g_c \gg 1$ for “piston” type deformation.
- $g_c \ll 1$ for dilations, translations and rotations.





Barnett, Cohen, Heller (PRL 2000, JPA 2000)

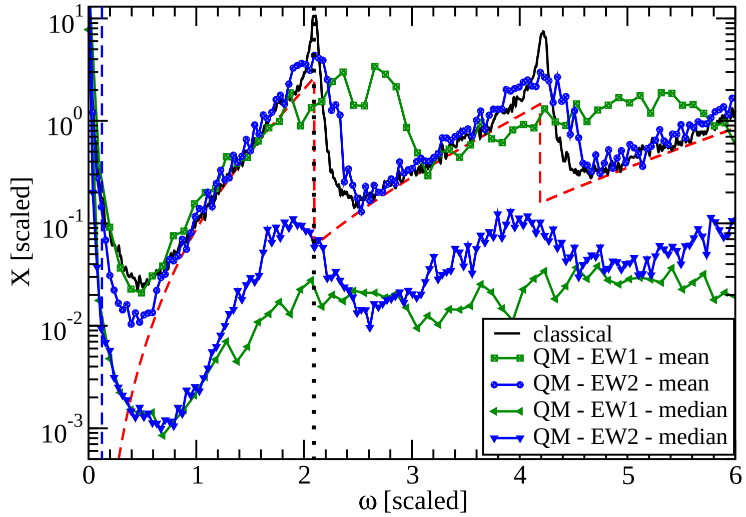
Heating of particles by "vibrating" a piston

$$\mathcal{H}_{\text{total}} \approx \Big[\mathcal{H}_0 + U\Big] + f(t)V$$

$$\mathcal{H} \;=\; \text{rectangular } (L_x \times L_y)$$

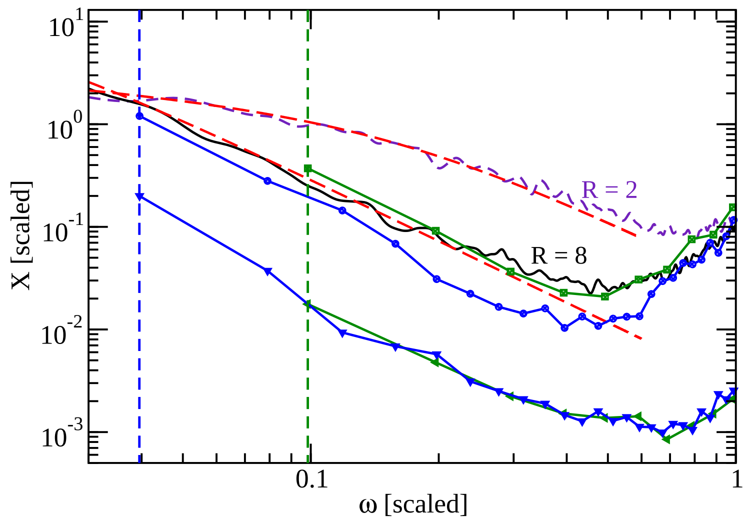
$$U \;=\; \text{deformation } (u = L/R)$$

$$\tilde{C}(\omega) \;=\; \text{FT } \langle V(t)V(0)\rangle$$



$$C(\omega \gg 1/t_{\text{L}}) = \frac{8}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x} \;\equiv\; C_\infty$$

$$\tilde{C}(\omega \ll 1/t_{\text{L}}) \approx C_\infty \times \left(\frac{1}{u}\right) \times \ln\left(\frac{2}{\omega t_R}\right)$$

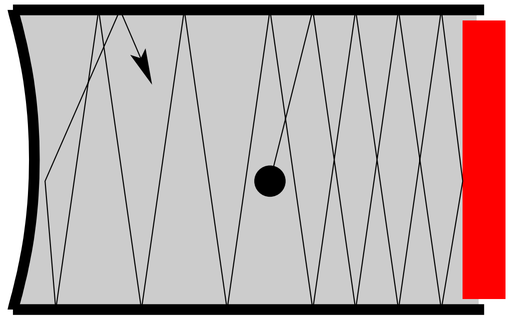


Kubo formula:

$$\boldsymbol{D} = \int_0^\infty \tilde{C}(\omega)\tilde{S}(\omega)d\omega \;=\; g_c\,\frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\,\overline{\dot{f}^2}$$

Stotland, Pecora, Cohen (2010)

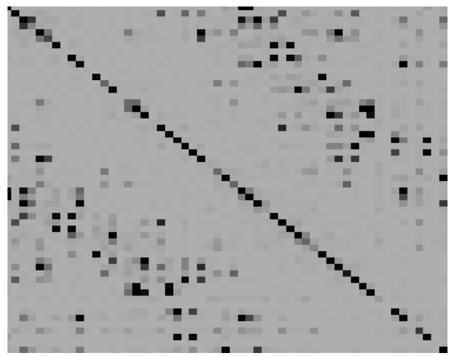
The sparsity of the perturbation matrix



The Hamiltonian in the $\boldsymbol{n} = (n_x, n_y)$ basis:

$$\mathcal{H} = \mathrm{diag}\{E_{\boldsymbol{n}}\} + u\{U_{\boldsymbol{nm}}\} + f(t)\{V_{\boldsymbol{nm}}\}$$

The matrix elements for the wall displacement:

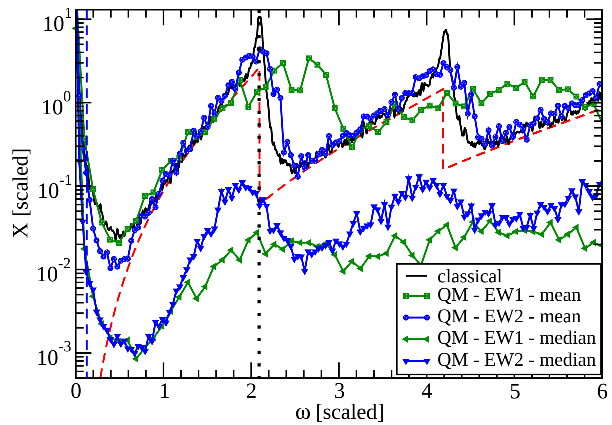


$$V_{\boldsymbol{nm}} \quad = \quad -\delta_{n_y,m_y} \times \frac{\pi^2}{\mathsf{M} L_x^3} n_x m_x \qquad \text{[sparse]}$$

The Hamiltonian in the E_n basis:

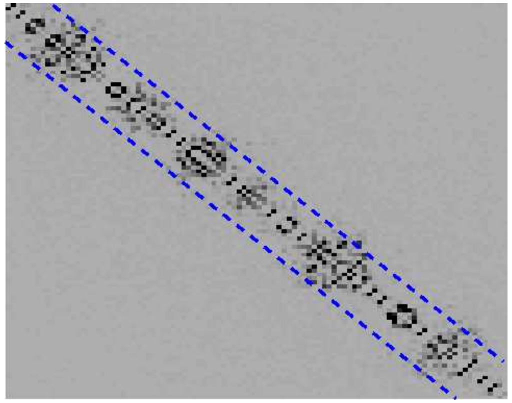
$$\mathcal{H} = \mathrm{diag}\{E_n\} + f(t)\{V_{nm}\}$$

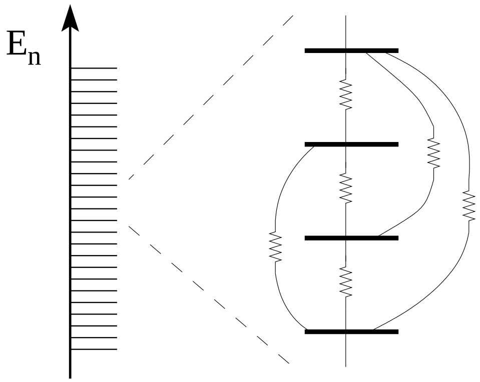
The Kubo formula (LRT):



$$\boldsymbol{D} \;=\; \pi \varrho_{\mathrm{E}} \, \langle\langle |V_{mn}|^2 \rangle\rangle_a \, \overline{\dot{f}^2} \;=\; g_c \boldsymbol{D}_0$$

$\{|V_{nm}|^2\}$ as a random matrix $\boldsymbol{X} = \{x\}$

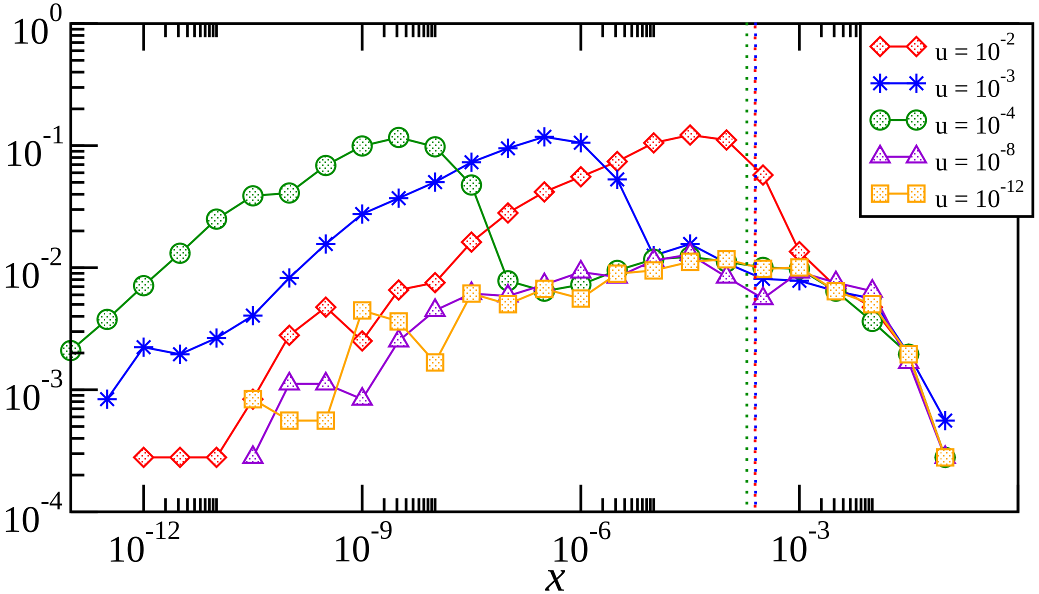




$$s[\boldsymbol{X}] \equiv \frac{\mathrm{PN}[\boldsymbol{X}]}{\mathrm{PN}[\boldsymbol{X}^{\mathrm{unf}}]} = \text{sparsity}$$

$$g_s[\boldsymbol{X}] \equiv \frac{\langle\langle \boldsymbol{X} \rangle\rangle_s}{\langle\langle \boldsymbol{X} \rangle\rangle_a} = \text{connectivity}$$

Histogram of x :



$x \sim \mathrm{LogNormal}$

For a random sparse matrix:

$$s,\ g_s \ll 1$$

For a uniform (along diagonals):

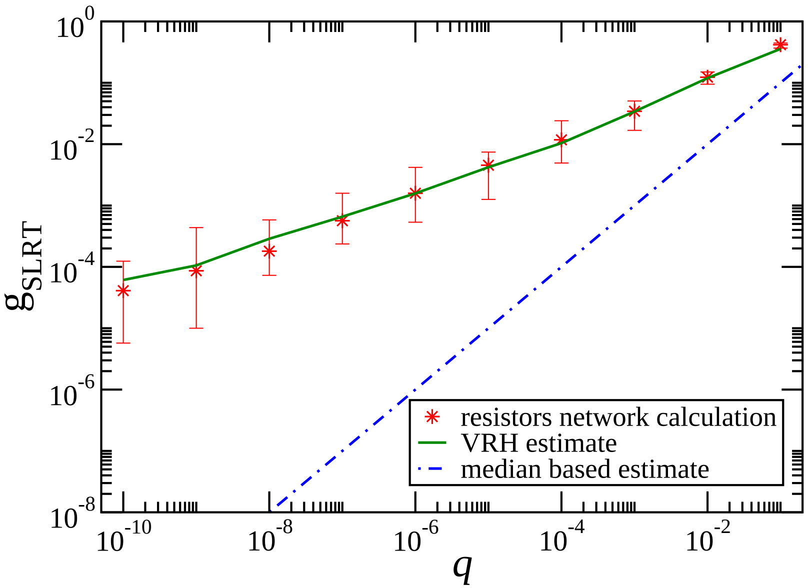
$$s = g_s = 1$$

For a Gaussian matrix:

$$s = 1/3, \quad g_s \sim 1$$

RMT modeling, generalized VRH approx scheme

- log-normal distribution q
- finite bandwidth b



$$s \;=\; q^2 \;=\; (\text{median}/\text{mean})^2$$

$$g_s \;\approx\; q\;\exp\left[2\sqrt{-\ln q\;\ln(b)}\right]$$

Digression: Generalized VRH

Definition of the typical matrix element for a range ω transition:

$$\left(\frac{\omega}{\Delta}\right) \mathrm{Prob}\Big(x > x_\omega\Big) \;\sim\; 1$$

In the standard-like case (ring with strong disorder):

$$x_\omega \;\approx\; v_\mathrm{F}^2\; \exp\left(\frac{\Delta_l}{|\omega|}\right)$$

[corresponding to a log-box distribution]

An example for the power spectrum of the driving:

$$\tilde{S}(\omega) \;\propto\; \exp\left(-\frac{|\omega|}{T}\right)$$

[here the temparature $T \Longleftrightarrow \omega_c$]

Generalized VRH estimate:

$$D_\mathrm{SLRT} \;\approx\; \int x_\omega\; \tilde{S}(\omega) d\omega$$

[should be contrasted with]

$$D_\mathrm{LRT} = \int \tilde{C}(\omega)\; \tilde{S}(\omega) d\omega$$

In the standard-like case (ring with strong disorder):

$$D_\mathrm{SLRT} \;\approx\; \int \exp\left(\frac{\Delta_l}{|\omega|}\right)\, \exp\left(-\frac{|\omega|}{T}\right)\; d\omega$$

The resistor network calculation

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

$$g_s \quad \equiv \quad \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}$$

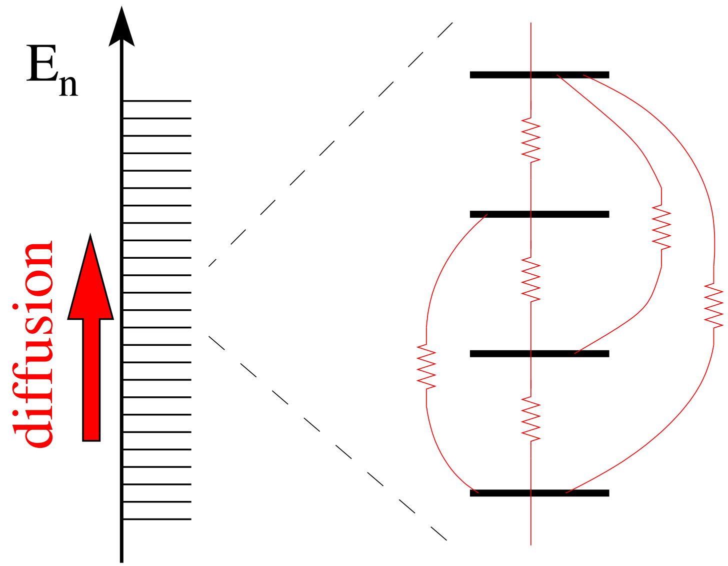
$$\boldsymbol{D} \quad = \quad \boldsymbol{G} \ \overline{\dot{f}^2}$$

LRT applies if the driven transitions are slower than the environmental relaxation

$$\boldsymbol{G}_{\rm LRT} \quad = \quad \pi\varrho \ \langle\langle |V_{nm}|^2 \rangle\rangle_a$$

$$\boldsymbol{G}_{\rm SLRT} \quad = \quad \pi\varrho \ \langle\langle |V_{nm}|^2 \rangle\rangle_s$$

$$\mathrm{g}_{nm} = 2\varrho^{-3} \ \frac{|V_{nm}|^2}{(E_n - E_m)^2} \ \delta_0(E_n - E_m)$$



$$\langle\langle |V_{nm}|^2 \rangle\rangle_s \equiv \text{inverse resistivity}$$

Digression: the Fermi golden rule picture

Master equation:

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{f}(t)\left\{\frac{iV_{nm}}{E_n - E_m}\right\}$$

$$\tilde{S}(\omega) \equiv \text{FT}\,\langle \dot{f}(t)\dot{f}(0)\rangle = 2\pi\overline{|\dot{f}|^2}\,\delta_0(E_n - E_m)$$

The FGR transition rate due to the low frequency noisy driving:

$$w_{nm} = \left|\frac{V_{nm}}{E_n - E_m}\right|^2 \tilde{S}(E_n - E_m) \equiv \pi\varrho^3\, \mathrm{g}_{nm}\, \overline{\dot{f}^2}$$

The LRT / SLRT formula

$$\boldsymbol{D} = \text{average}\left[\frac{1}{2}\sum_n (E_n - E_m)^2\, w_{nm}\right] = \pi\varrho\,\langle\langle |V_{nm}|^2\rangle\rangle \times \overline{\dot{f}^2} \equiv \boldsymbol{G}\,\overline{\dot{f}^2}$$

Digression: random walk and the calculation of the diffusion coefficient

$w_{nm} =$ probability to hop from m to n per step.

$$\mathrm{Var}(n) = \sum_n [w_{nm} t]\,(n-m)^2 \equiv 2Dt$$

For n.n. hopping with rate w we get $D = w$.

The continuity equation:

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n$$

Fick's law:

$$J_n = -D\frac{\partial}{\partial n} p_n$$

The diffusion equation:

$$\frac{\partial p_n}{\partial t} = D\frac{\partial^2}{\partial n^2} p_n$$

If we have a sample of length N then

$$J = -\frac{D}{N} \times [p_N - p_0]$$

$$D/N = \text{inverse resistance of the chain}$$

If the w are not the same:

$$\frac{D}{N} = \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}$$

Hence, for n.n. hopping

$$D = \langle\langle w \rangle\rangle_{\text{harmonic}}$$

FGR: $w_{nm} \sim |V_{nm}|^2$

$$D = \langle\langle |V_{nm}|^2 \rangle\rangle$$

SLRT *vs* LRT

$$\mathcal{H}_{\text{total}} \approx \mathcal{H} + f(t)V$$

$$\tilde{C}(\omega) = \text{FT}\,\langle V(t)V(0)\rangle$$

$$\tilde{S}(\omega) = \text{FT}\,\langle \dot{f}(t)\dot{f}(0)\rangle$$

Kubo formula:

$$\boldsymbol{D} = \int \tilde{C}(\omega)\tilde{S}(\omega)d\omega$$

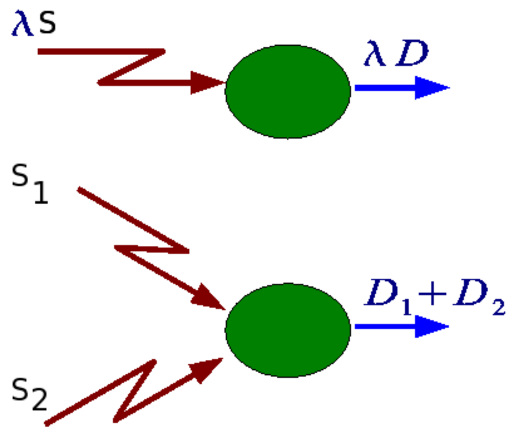
SLRT example:

$$\boldsymbol{D} = \left[\int R(\omega)\left[\tilde{S}(\omega)\right]^{-1} d\omega\right]^{-1}$$

Linear response implies

$$\tilde{S}(\omega) \mapsto \lambda\tilde{S}(\omega) \quad\Longrightarrow\quad \boldsymbol{D} \mapsto \lambda\boldsymbol{D}$$

$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \quad\Longrightarrow\quad \boldsymbol{D} \mapsto \sum_i \boldsymbol{D}_i$$



The weak quantum chaos regime

Consider $\mathcal{H}(u)$, where u is a control parameter.

Generically there are 3 parametric regimes [1]:

- First order perturbation theory regime
- Wigner / Fermi-Golden-Rule / Kubo regime
- Non-perturbative / SC / Lyapunov regime

Deforming a chaotic billiard [2,3]

The Wigner regime: $u_c < u < u_b$

$u_c = \hbar^{3/2}$ mixing of levels starts

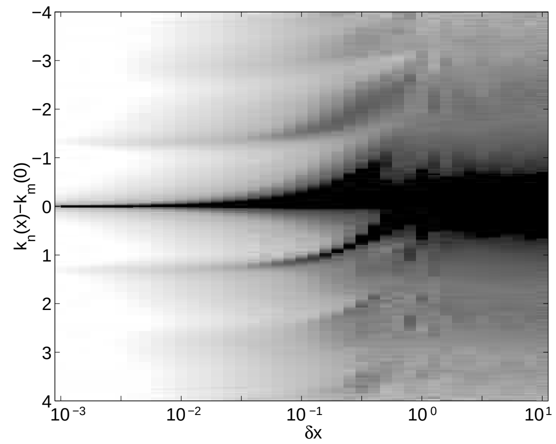
$u_b = \hbar^{1}$ mixing saturates ($\sim$bandwidth)

Deforming a rectangular billiard [4]

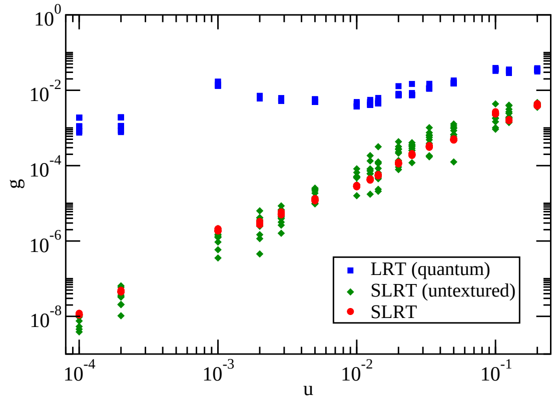
The WQC regime: $u_c < u < u_s$

$u_c = \hbar^{2}$ mixing of levels starts

$u_s = \hbar^{1/2}$ mixing saturates ($g_s \sim 1$)



$\#$ mixed levels $\approx (u/u_c)^2$



Mixing is non-uniform

[1] Cohen (PRL 1999, Annals 2000)
[2] Cohen and Heller (PRL 2000)
[3] Cohen, Barnett, Heller (PRE 2001)
[4] Stotland, Pecora, Cohen (2010)

Estimates for an experiment

Consider ^{77}Rb atoms at $T = 0.1\,\mu$K $\quad\rightsquigarrow\quad$ $\lambda_E = 1\,\mu$m

Linear size of the trap $L = 10\mu$m $\quad\rightsquigarrow\quad$ $h = \lambda_E/L = 0.01$,

SLRT suppression factor for $u \sim 10\%$ deformation is $g_s \sim 0.1$

- Ballistic frequency $\omega_L \approx 220\,$Hz
- Lyapunov frequency $\omega_R \approx 70\,$Hz, $\quad$ Driving $\omega_c \sim \omega_R$
- Level spacing $\omega_0 \approx 7.5\,$Hz

FGR condition: $\quad D/\omega_0^3 < (\omega_c/\omega_0)^{\mathrm{power}}$, $\quad$ power$= 2, 3$

Measurability condition: $\quad D/(T^2\omega_L) > 10^{-3}$

Conclusions

(*) Wigner ($\sim$ 1955):
The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. "weak quantum chaos" $\Longrightarrow$ (log-wide distribution).
2. The heating process $\sim$ a percolation problem.
3. Resistors network calculation to get G_{SLRT}.
4. Generalization of the VRH estimate
5. SLRT is essential whenever the distribution of matrix elements is wide ("sparsity") or if the matrix has "texture".