Semi-linear response of energy absorption

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Collaborations:

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Rangga Budoyo (Wesleyan)
Tal Peer (BGU)
Nir Davidson (Weizmann)

Discussions:

Yuval Gefen (Weizmann)
Shmuel Fishman (Technion)

\$ISF, \$GIF, \$DIP, \$BSF

Diffusion and Energy absorption

Driven chaotic system with Hamiltonian $\mathcal{H}(X(t))$

X = some control parameter

$\dot{X}$ = rate of the (noisy) driving

$\rightsquigarrow$ diffusion in energy space:

$$\boldsymbol{D} = \boldsymbol{G}_{\text{diffusion}} \ \overline{\dot{X}^2}$$

$\rightsquigarrow$ energy absorption:

$$\dot{\boldsymbol{E}} = \boldsymbol{G}_{\text{absorption}} \ \overline{\dot{X}^2}$$

[Ott, Brown, Grebogi, Wilkinson, Jarzynski, D.C.]

There is a dissipation-diffusion relation.

In the canonical case $\dot{E} = D/T$.

Below we use for G scaled units.

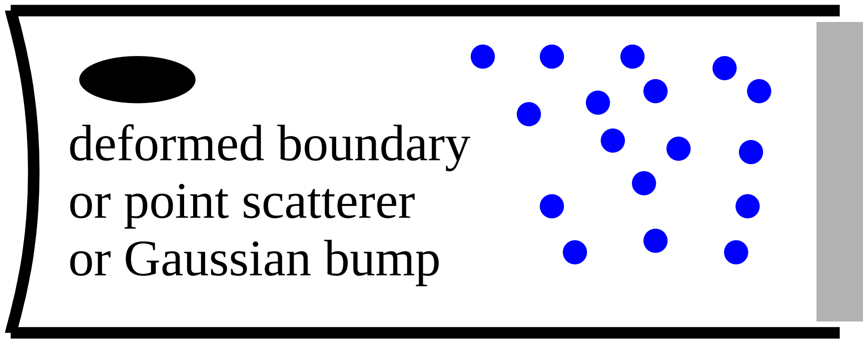
Models

$$\mathcal{H} = \{E_n\} - X(t)\{V_{nm}\}$$

with:

Stotland

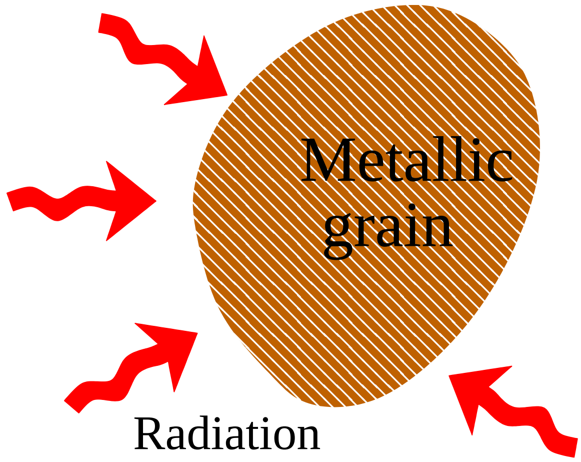
Davidson



with:

Wilkinson

Mehlig



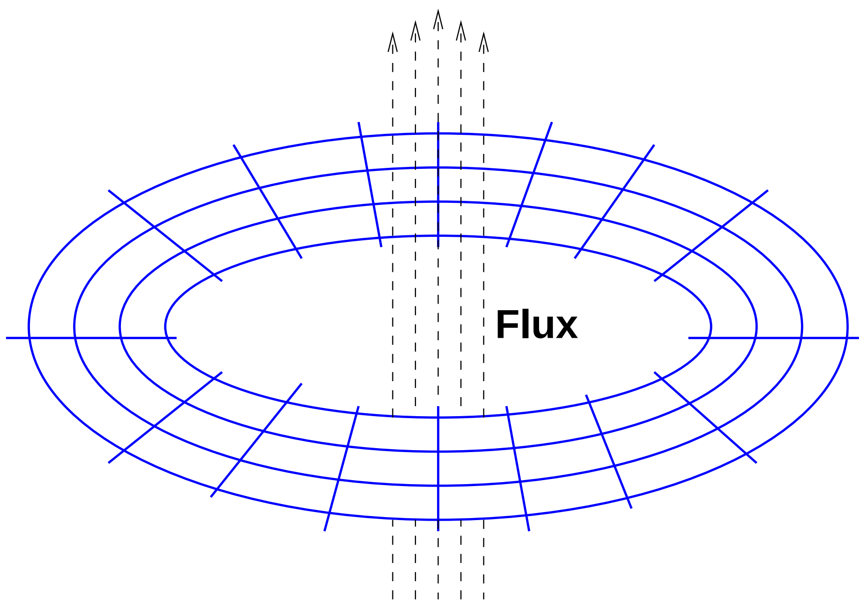
with:

Stotland

Budoyo

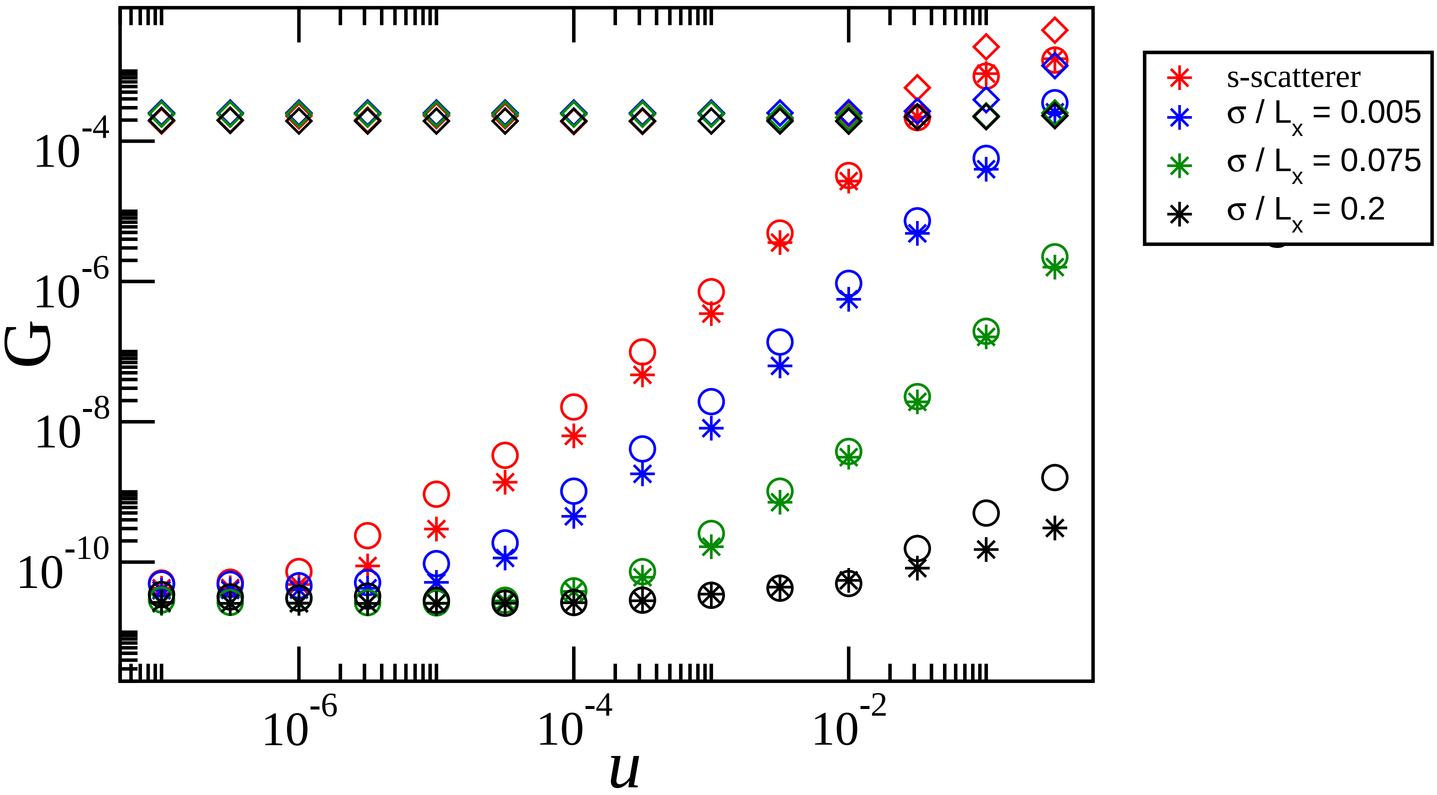
Peer

Kottos

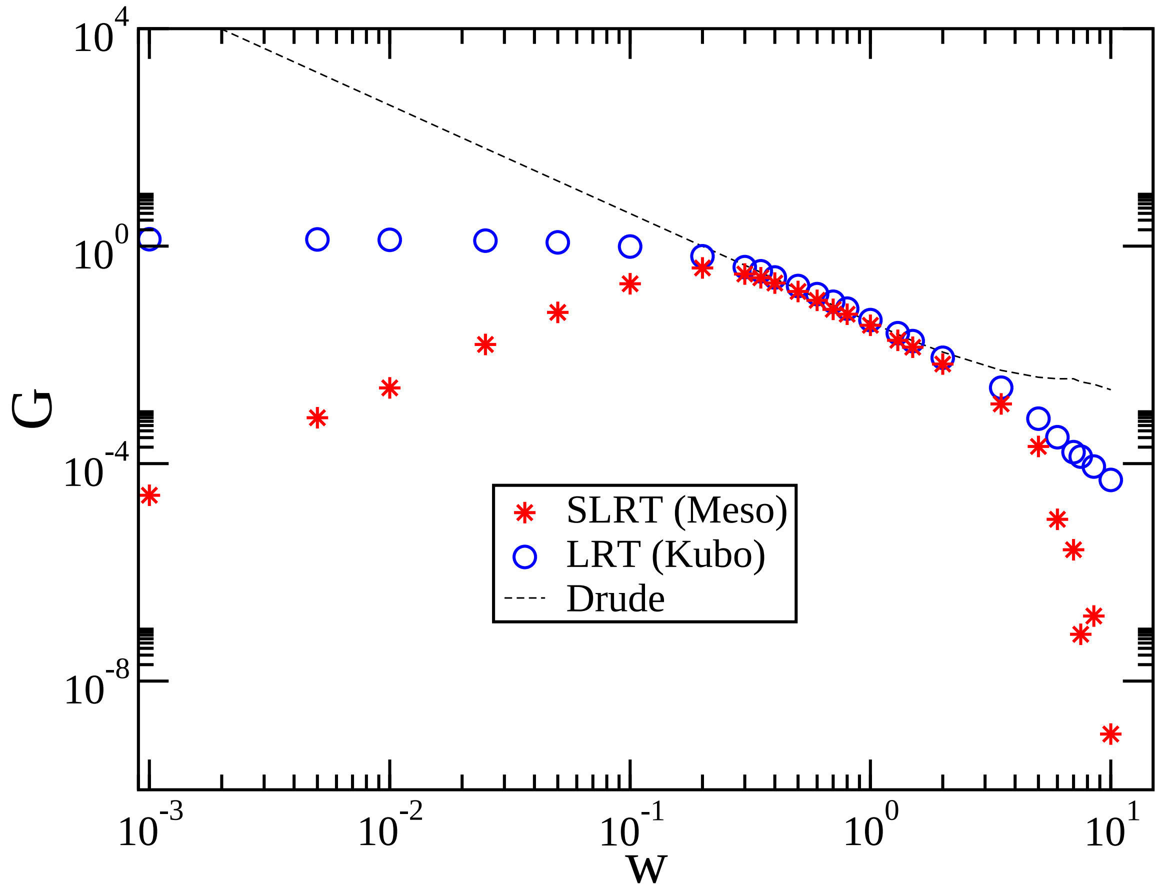


Some results

Cold atoms in vibrating traps:



Metallic rings driven by EMF:

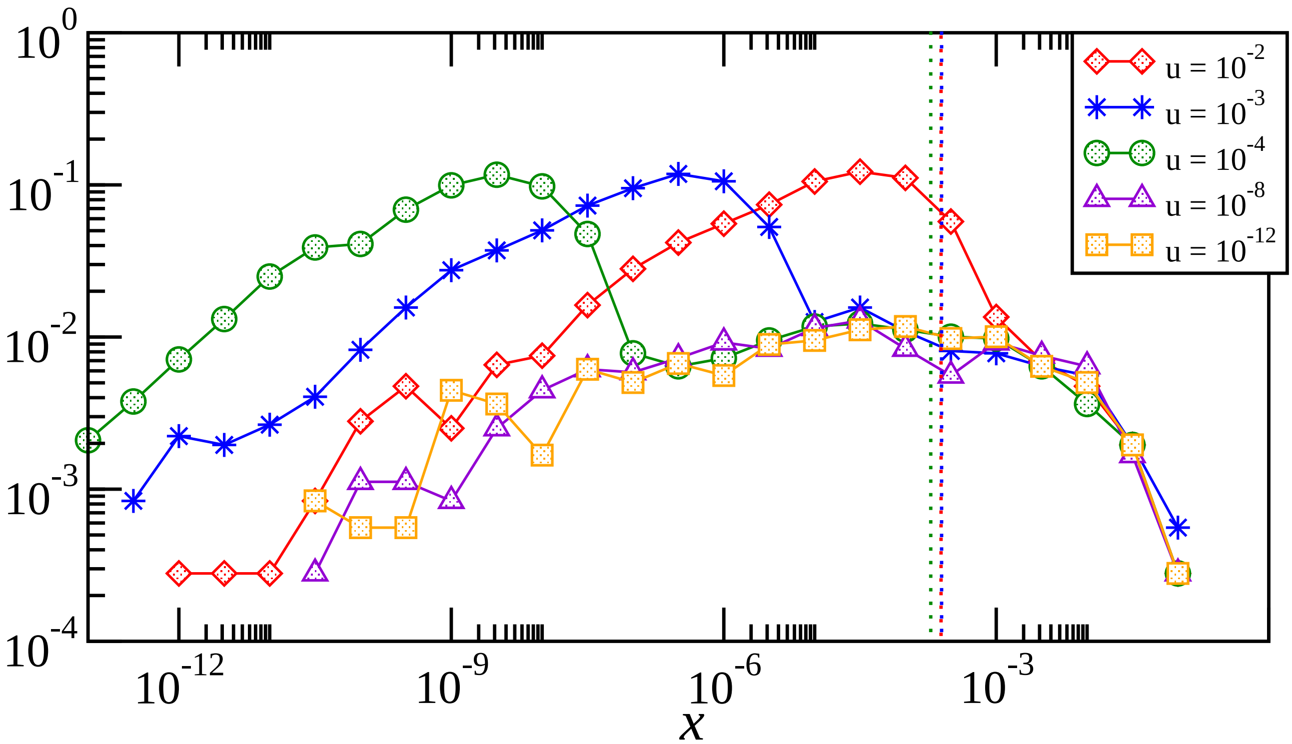


Digression: size distribution

Given a matrix that looks random $\{V_{nm}\}$,

Consider the *size distribution* of the elements.

Histogram of $\log(x)$ where $x = |V_{nm}|^2$



Algebraic average: $\langle\langle x \rangle\rangle_a = \langle x \rangle$

Harmonic average: $\langle\langle x \rangle\rangle_h = [\langle 1/x \rangle]^{-1}$

Geometric average: $\langle\langle x \rangle\rangle_g = \exp[\langle \log x \rangle]$

$$\langle\langle x \rangle\rangle_h \quad \ll \quad \langle\langle x \rangle\rangle_g \quad \ll \quad \langle\langle x \rangle\rangle_a$$

Digression: random walk

$w_{nm} =$ probability to hop from m to n per step.

$$\mathrm{Var}(n) \;=\; \sum_n [w_{nm} t]\,(n-m)^2 \;\equiv\; 2Dt$$

For n.n. hopping with rate w we get $D = w$.

The diffusion equation:

$$\frac{\partial p_n}{\partial t} \;=\; -\frac{\partial}{\partial n} J_n \;=\; D\frac{\partial^2}{\partial n^2} p_n$$

Fick's law:

$$J_n \;=\; -D\frac{\partial}{\partial n} p_n$$

If we have a sample of length N then

$$J \;=\; -\frac{D}{N} \times [p_N - p_0]$$

$D/N \;=\;$ inverse resistance of the chain

If the w are not the same:

$$\frac{D}{N} \;=\; \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}$$

Hence

$$D \;=\; \langle\langle w \rangle\rangle_h \qquad \text{for n.n. hopping}$$

Digression: Fermi Golden rule

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - X(t)\Big\{V_{nm}\Big\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{X}(t)\left\{\frac{iV_{nm}}{E_n - E_m}\right\}$$

The FGR transition rate for $\omega \sim 0$ driving:

$$w_{nm} \;=\; 2\pi \left|\frac{V_{nm}}{E_n - E_m}\right|^2 \overline{|\dot{X}|^2}\; \delta_\Gamma(E_n - E_m)$$

Note that the spectral content of the driving is

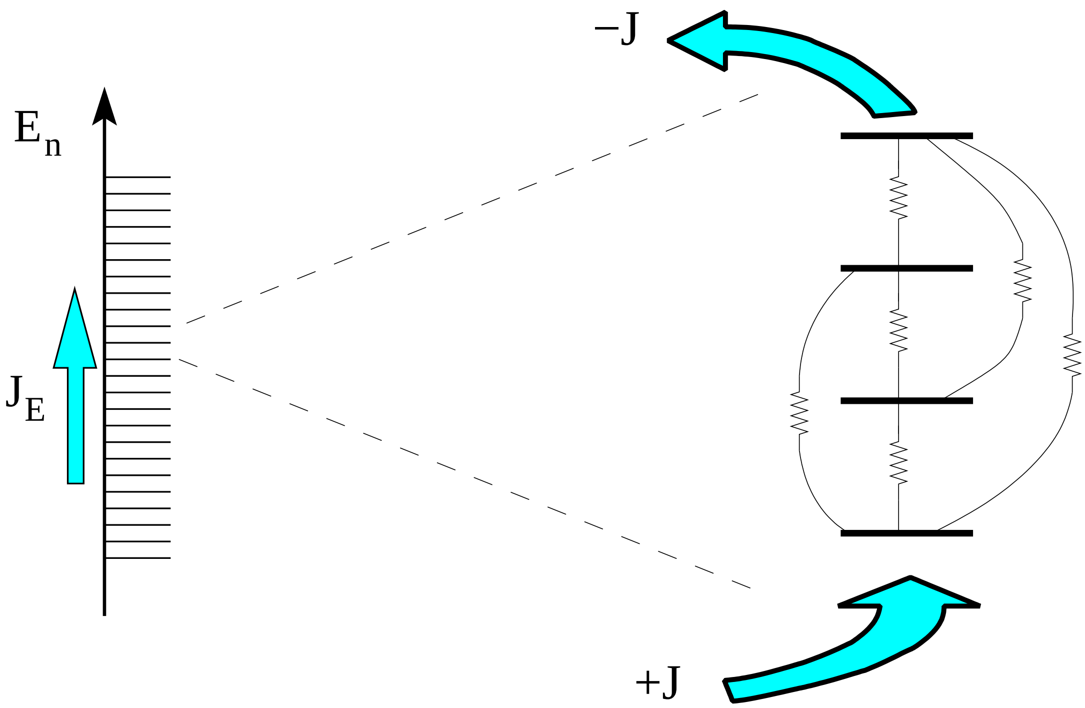
$$\tilde{S}(\omega) \;=\; \overline{|\dot{X}|^2}\; \delta_\Gamma(\omega - (E_n - E_m))$$

Semi Linear Response Theory (SLRT)

$$\mathcal{H} = \{E_n\} - X(t)\{V_{nm}\}$$

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

$$w_{nm} \quad = \quad \text{const} \times \mathrm{g}_{nm} \times \overline{\dot{X}^2}$$



$$\mathrm{g}_{nm} = 2\varrho^{-3} \ \frac{|V_{nm}|^2}{(E_n - E_m)^2} \ \delta_\Gamma(E_n - E_m)$$

$$\langle\langle |V_{mn}|^2 \rangle\rangle \ \equiv \ \text{inverse resistivity of the network}$$

$$\boldsymbol{D} \quad = \quad \pi\varrho \ \langle\langle |V_{mn}|^2 \rangle\rangle \times \overline{\dot{X}^2} \quad \equiv \quad \boldsymbol{G} \ \overline{\dot{X}^2}$$

Example: cold atoms in vibrating trap

The Hamiltonian in the $\boldsymbol{n} = (n_x, n_y)$ basis:

$$\mathcal{H} = \mathrm{diag}\{E_{\boldsymbol{n}}\} + u\{U_{\boldsymbol{nm}}\} + f(t)\{V_{\boldsymbol{nm}}\}$$

The matrix elements for the wall displacement:

$$V_{\boldsymbol{nm}} \quad = \quad -\delta_{n_y,m_y} \times \frac{\pi^2}{\mathsf{M} L_x^3} n_x m_x$$

The Hamiltonian in the E_n basis:

$$\mathcal{H} = \mathrm{diag}\{E_n\} + f(t)\{V_{nm}\}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_a \approx \frac{\mathsf{M} v_{\mathrm{E}}^3}{2\pi L_x^2 L_y}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_g \approx \frac{4\mathsf{M}^2 v_{\mathrm{E}}^4}{L_x^3 L_y \omega_c^2} \exp\left[-\mathsf{M}^2 v_{\mathrm{E}}^2 (\sigma_x^2 + \sigma_y^2)\right] \times u^2$$

The SLRT result:

$$G_{\mathrm{SLRT}} \quad = \quad q \ \exp\left[2\sqrt{-\ln q}\right] \times G_{\mathrm{LRT}}$$

SLRT vs LRT

$$X \quad = \quad \text{some control parameter}$$

$$\dot{X} \quad = \quad \text{rate of the (noisy) driving}$$

The definition of the "conductance":

$$\boldsymbol{D} \quad = \quad \boldsymbol{G}\,\overline{\dot{X}^2}$$

LRT implies

$$\boldsymbol{D} \quad = \quad \int G(\omega)|\dot{X}_\omega|^2\,d\omega \quad = \quad \int G(\omega)\tilde{S}(\omega)\,d\omega$$

Within the framework of LRT

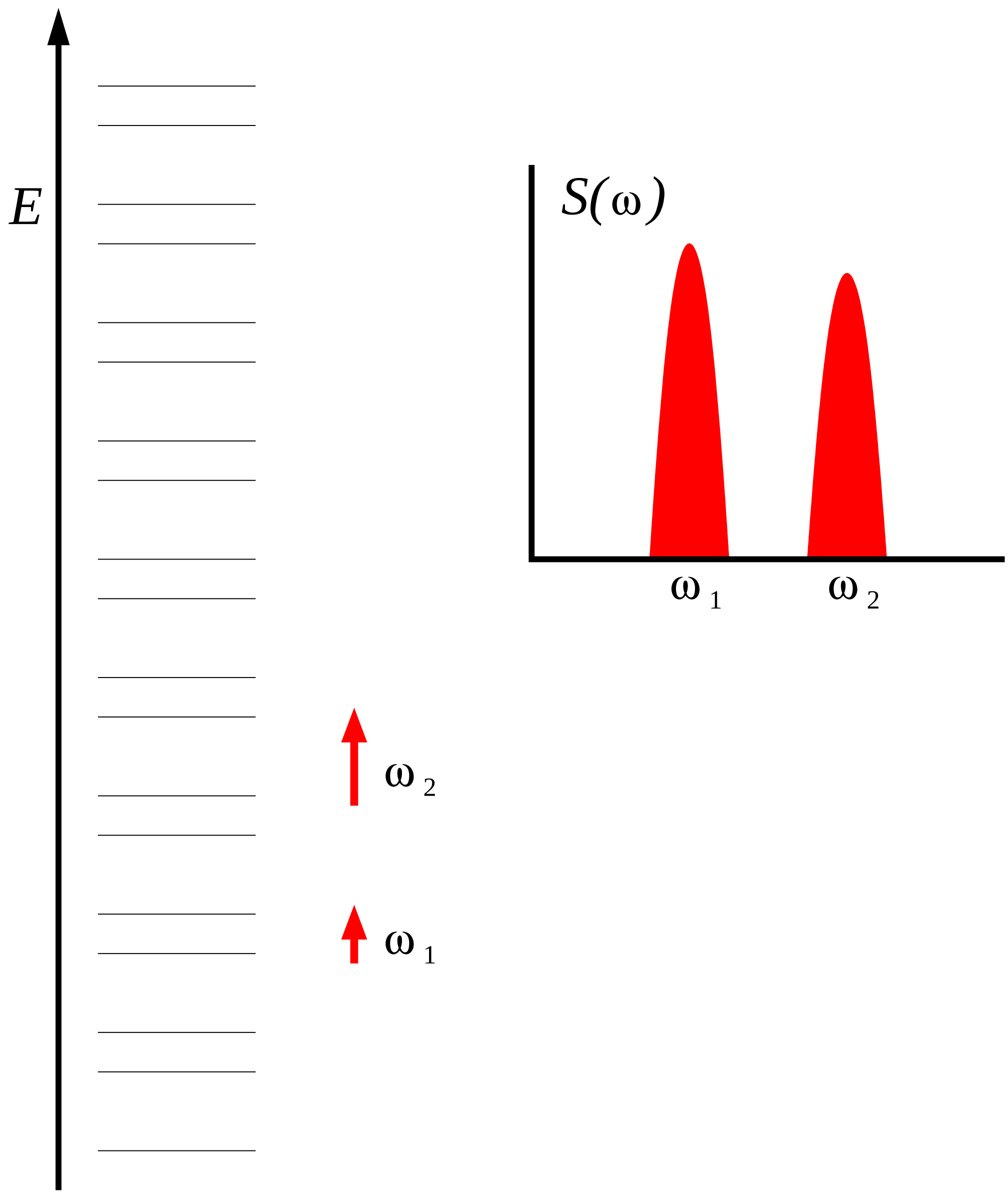
$$\tilde{S}(\omega) \mapsto \lambda\tilde{S}(\omega) \qquad \Longrightarrow \qquad \boldsymbol{D} \mapsto \lambda\boldsymbol{D}$$

$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \qquad \Longrightarrow \qquad \boldsymbol{D} \mapsto \sum_i \boldsymbol{D}_i$$

But there are circumstance such that e.g.

$$\boldsymbol{D} \quad = \quad \left[\int R(\omega)\left[\tilde{S}(\omega)\right]^{-1} d\omega\right]^{-1}$$

Simplest illustration



$$D \gg D_1 + D_2$$

Example: energy absorption by metallic grains

Linear response theory:

$$\boldsymbol{D} = \sigma^2 \hbar \varrho \int_0^\infty \mathrm{d}\omega\, \omega^2\, R_2(\hbar\omega) \tilde{S}(\omega)$$

Semi-linear response theory:

$$\boldsymbol{D} \;=\; \frac{\sigma^2}{(\varrho\hbar)^3} \left[\int \frac{\mathrm{d}\mathbf{x}\, \mathrm{e}^{-\mathbf{x}^2/2}}{(2\pi)^{N/2} \mathbf{x}^2} \right]^{-1} \left[\int_0^\infty \mathrm{d}\omega \frac{P_2(\varrho\hbar\omega)}{\tilde{S}(\omega)} \right]^{-1}$$

Level spacing statistics:

$$P_2(s) \approx a_\beta s^\beta \exp(-c_\beta s^2) \qquad \text{with } \beta = 1, 2, 4$$

The LRT result of Gorkov and Eliashberg:

$$\boldsymbol{G} \;=\; C_\beta \sigma^2 (\hbar\varrho)^{\beta+1}\, T^{\beta+2}$$

Our SLRT result (large s statistics!):

$$\boldsymbol{G} \;=\; \frac{\sigma^2}{2\hbar\varrho} \frac{1}{(\hbar\varrho\omega_0)^{\beta-1}} \exp\left[-\frac{1}{\pi(\hbar\varrho T)^2} \right]$$

The conductance of small mesoscopic disordered rings

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The model

Non interacting "spinless" electrons in a ring.

$$\mathcal{H}(r, p; \Phi(t))$$

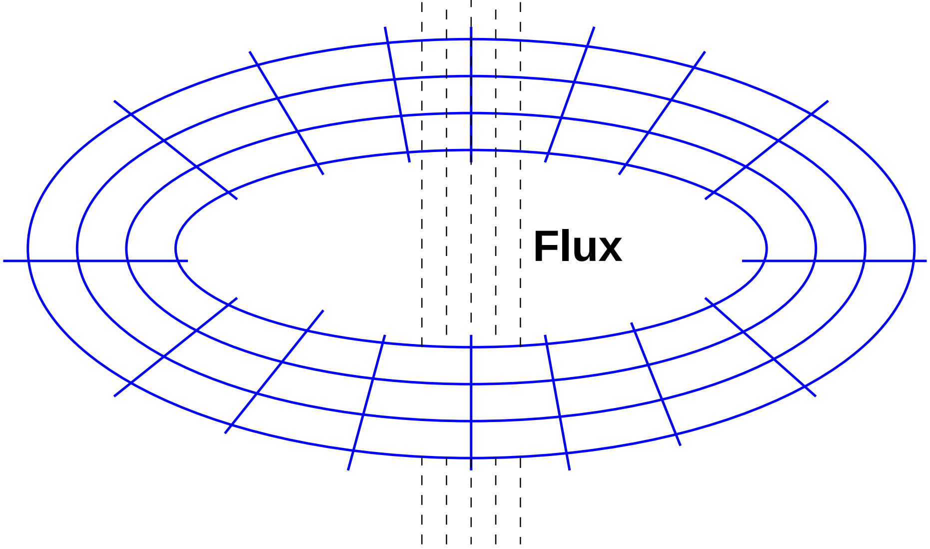
$$-\dot{\Phi} \quad = \quad \text{electro motive force (RMS)}$$

$$\boldsymbol{G}\,\dot{\Phi}^2 \quad = \quad \text{rate of energy absorption}$$

$$\boldsymbol{G} = \pi \left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle$$

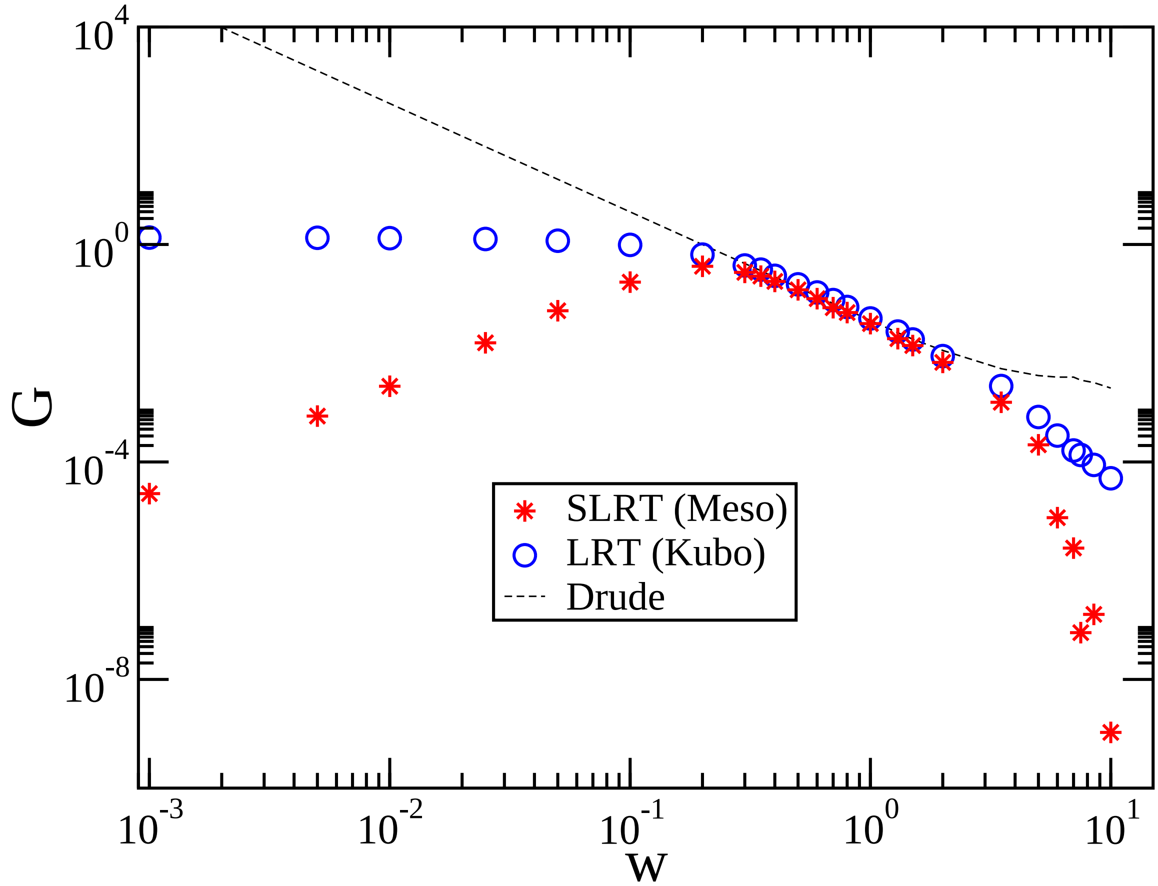
$$\langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{harmonic}} \ll \langle\langle |v_{mn}|^2 \rangle\rangle \ll \langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{algebraic}}$$

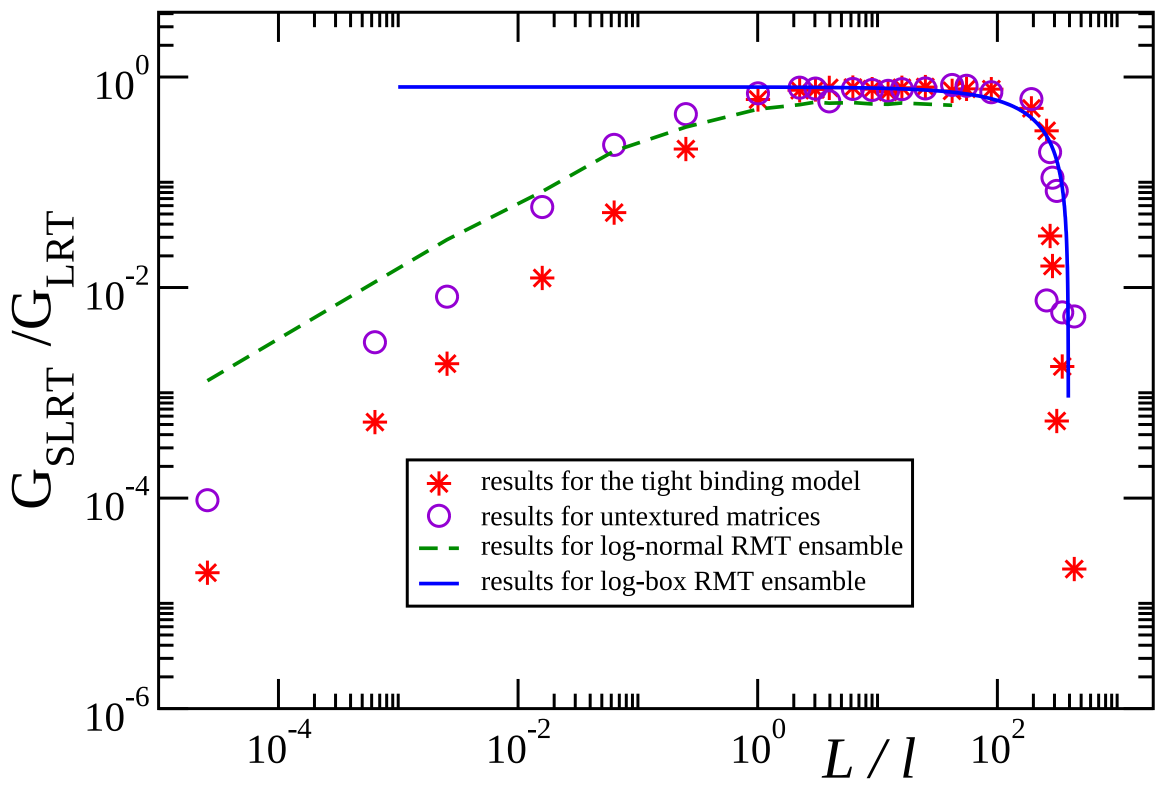
$\mathcal{M}$ mode ring of length L with disorder W



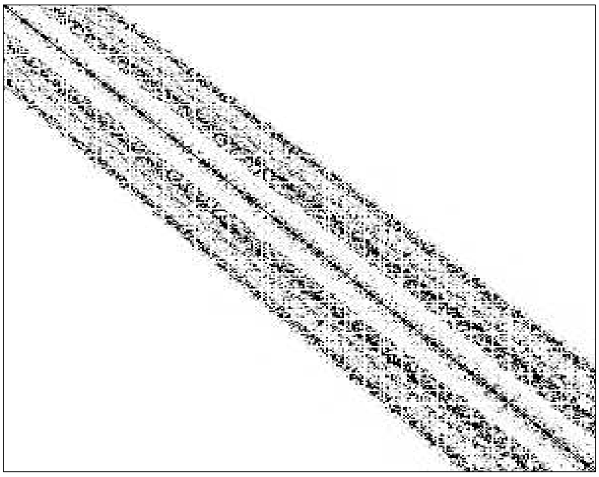
Numerical Results

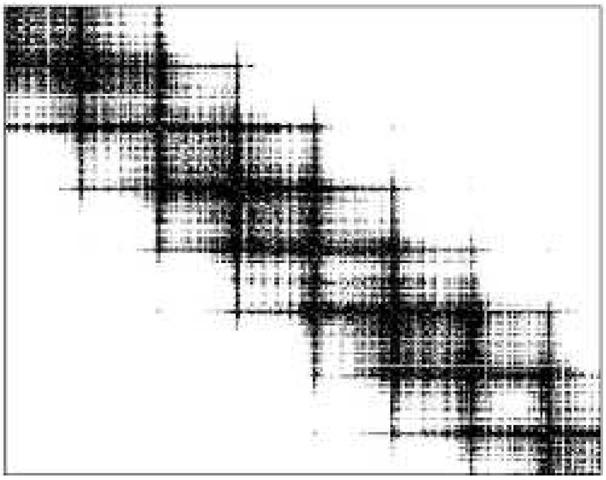
Regimes: ballistic; diffusive; localizaion





Linear response theory (LRT)





$$\mathcal{H} = \{E_n\} - \frac{e}{L}\Phi(t)\{v_{nm}\}$$

$$\boldsymbol{G} \;=\; \pi\left(\frac{e}{L}\right)^2 \sum_{n,m} |v_{mn}|^2 \; \delta_T(E_n - E_F) \; \delta_\Gamma(E_m - E_n)$$

$$\boldsymbol{G} \;=\; \pi\left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{algebraic}}$$

applies if

EMF driven transitions $\ll$ relaxation

otherwise

connected sequences of transitions are essential.

leading to

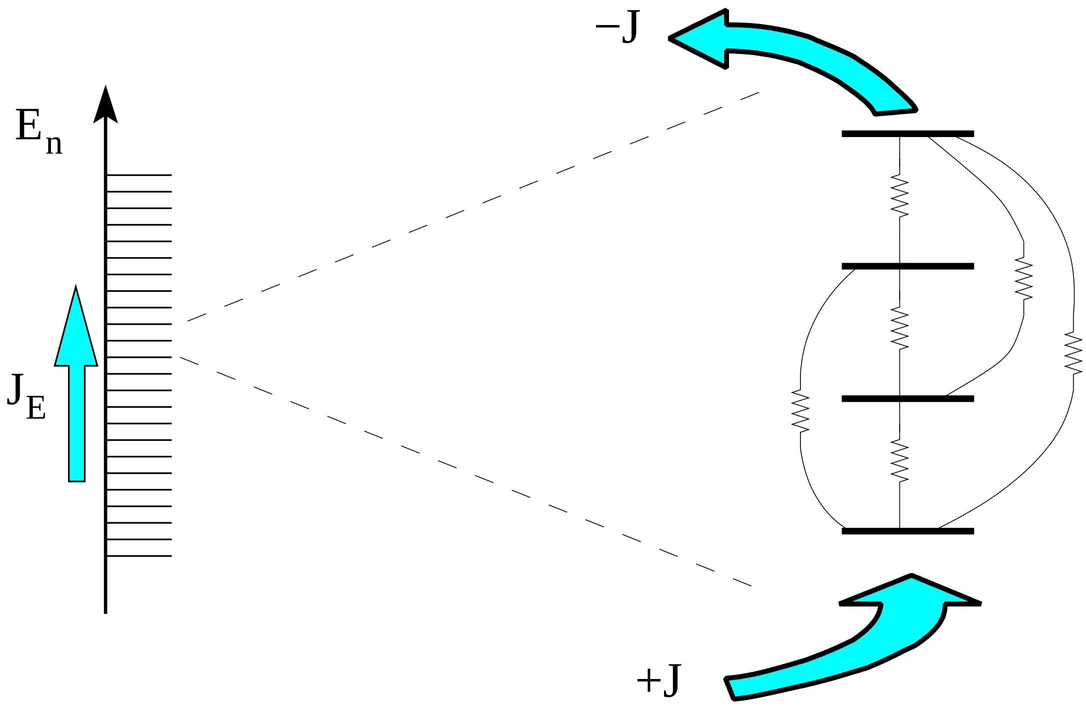
Semi Linear Response Theory (SLRT)

Semi Linear Response Theory (SLRT)

$$\mathcal{H} = \{E_n\} - \frac{e}{L}\Phi(t)\{v_{nm}\}$$

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

$$w_{nm} \quad = \quad \text{const} \times \mathrm{g}_{nm} \times \text{EMF}^2$$

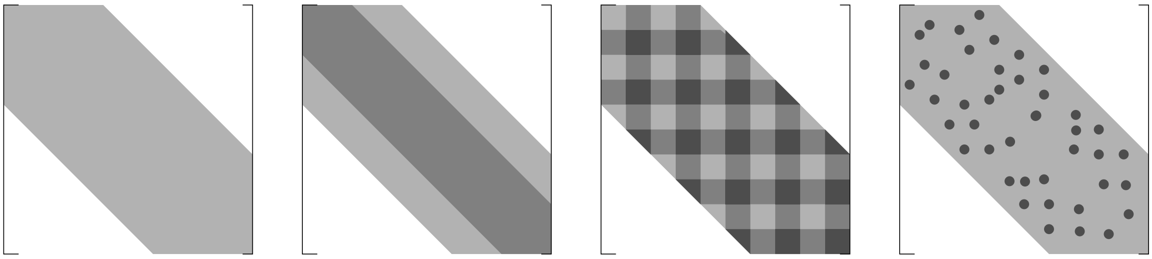


$$\mathrm{g}_{nm} = 2\varrho_{\mathrm{F}}^{-3} \; \frac{|v_{nm}|^2}{(E_n - E_m)^2} \; \delta_{\Gamma}(E_n - E_m)$$

$$\langle\langle |v_{mn}|^2 \rangle\rangle \;\equiv\; \text{inverse resistivity of the network}$$

$$\boldsymbol{G} \;=\; \pi \left(\frac{e}{L}\right)^2 \text{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle$$

Bandprofile, sparsity and texture



$$\boldsymbol{G} = \pi \left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \langle\langle |v_{mn}|^2 \rangle\rangle$$

$$\langle\langle |v_{mn}|^2 \rangle\rangle \equiv \text{inverse resistivity of the network}$$

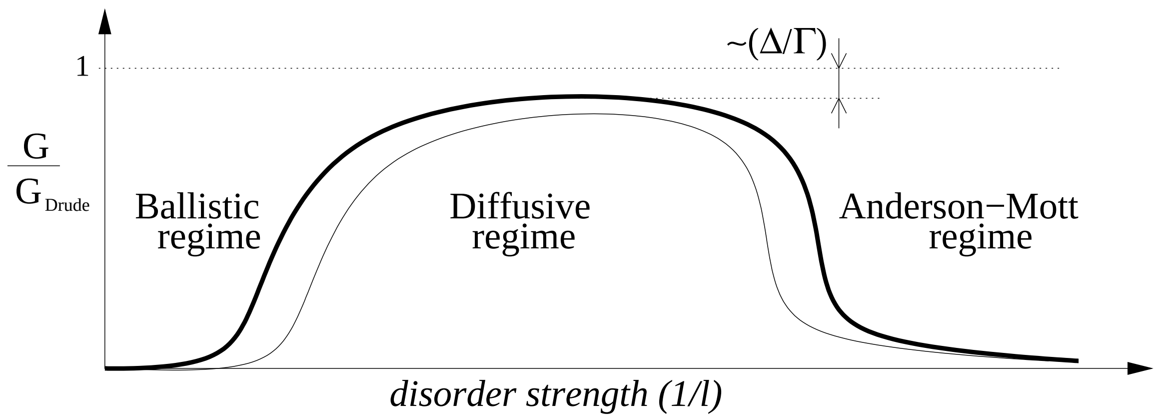
Bounds:

$$\langle\langle |v_{mn}|^2 \rangle\rangle_{\text{harmonic}} \ll \langle\langle |v_{mn}|^2 \rangle\rangle \ll \langle\langle |v_{mn}|^2 \rangle\rangle_{\text{algebraic}}$$

Analytical estimates:

- Mixed average scheme
- Variable range hopping scheme

Conductance versus disorder



Naive expectation (assuming $\Gamma > \Delta$):

$$G = \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{\ell}{L} + \mathcal{O}\left(\frac{\Delta}{\Gamma}\right)$$

L = perimeter of the ring

ℓ = mean free path $\propto W^2$

ℓ_∞ = localization length $\approx \mathcal{M}\ell$

Ballistic regime: $L \ll \ell$

Diffusive regime: $\ell \ll L \ll \ell_\infty$

Anderson regime: $\ell_\infty \ll L$

Strategy of analysis

Given W...

Characterization of the eigenstates:

- participation ratio (PR)

Characterization of v_{nm} and RMT modeling

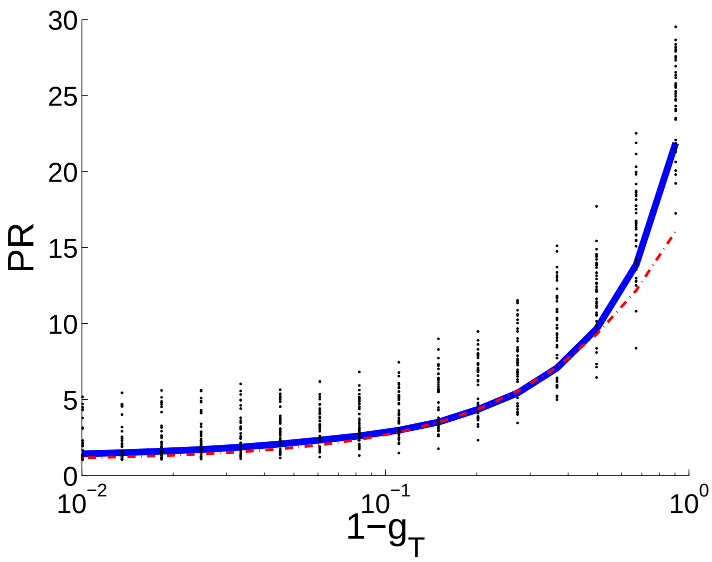
- bandwidth
- sparsity (p)
- texture

Approximation schemes for G

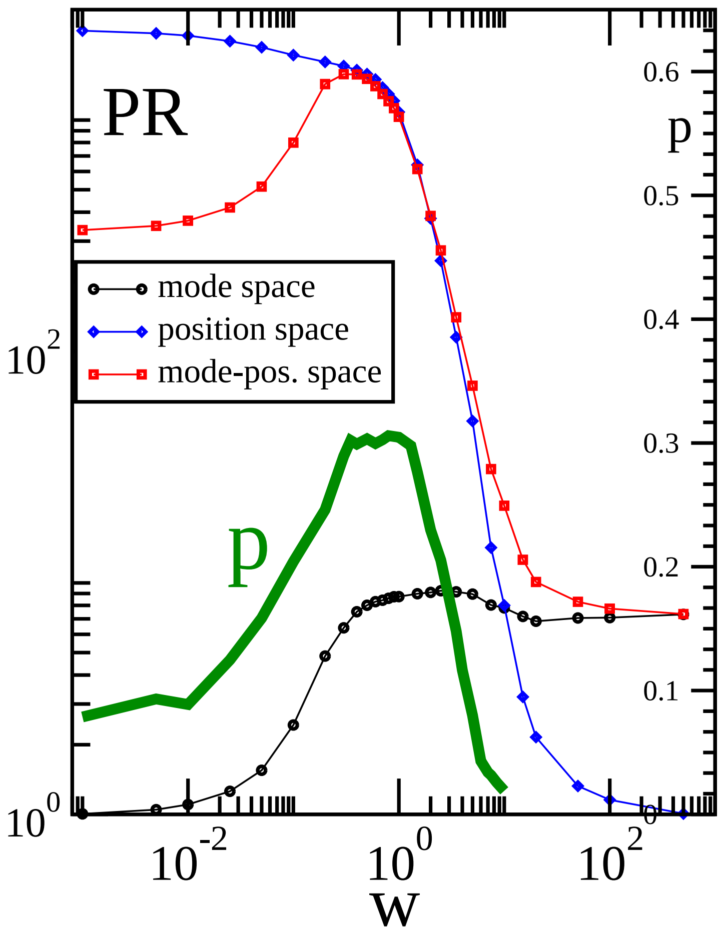
- Mixed average
- Variable range hopping estimate

Ergodicity of the eigenstates

- Weak disorder (ballistic rings):
  Wavefunctions are localized in mode space.
- Strong disorder (Anderson localization):
  Wavefunctions are localized in real space.



The PR of eigenstates of a ring with a single scatterer. The horizontal axis is the reflection of the scatterer.



The PR of eigenstates of a ring with disorder. The horizontal axis is W.

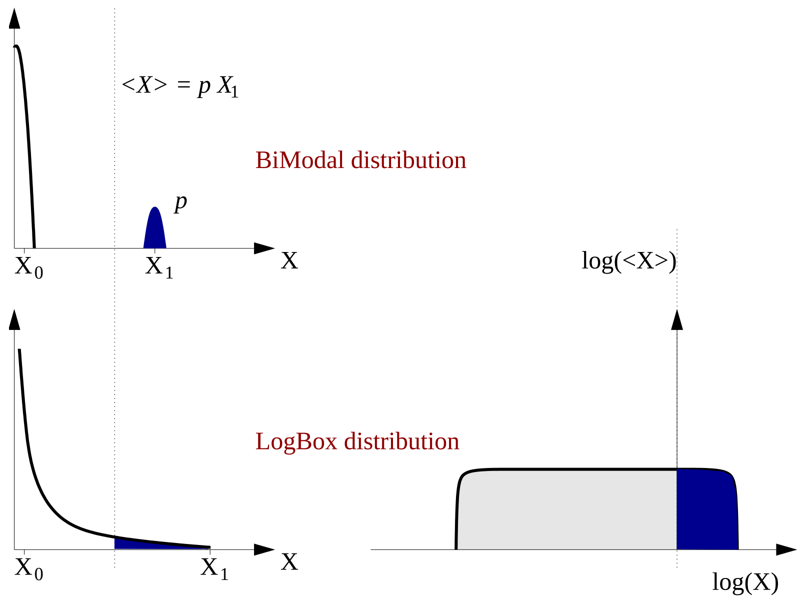
The sparsity (p) of the perturbation matrix is related to the ergodicity of the eigenstates.

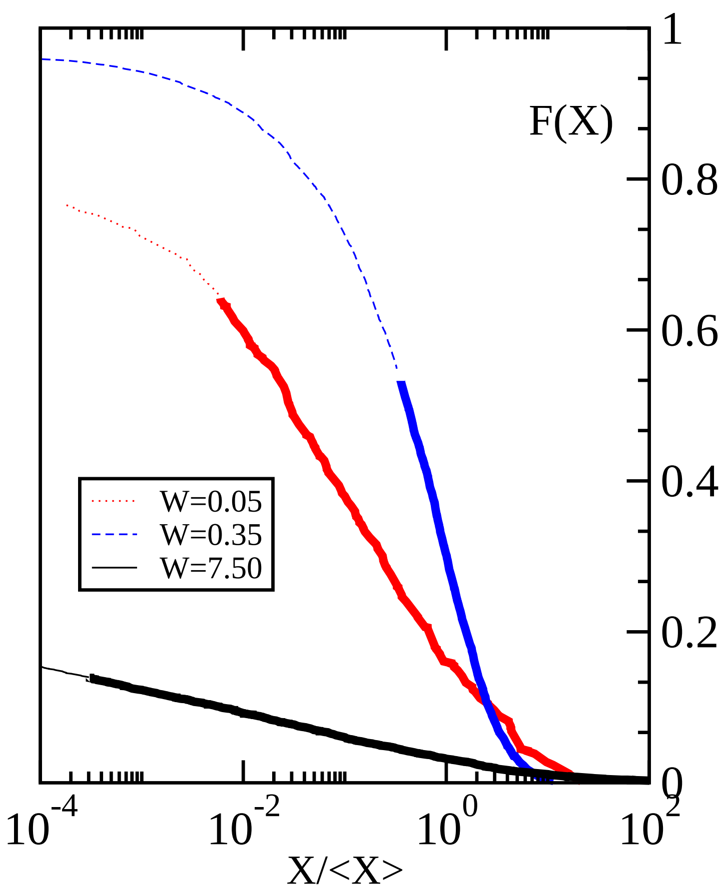
$\{|v_{nm}|^2\}$ **as a random matrix** $\{X\}$

The fraction of "large" elements:

$p \equiv \mathrm{F}(\langle X \rangle)$

Sparsity: $p \ll 1$.





Histograms of X:

Ballistic:

$X \sim \mathrm{LogNormal}$

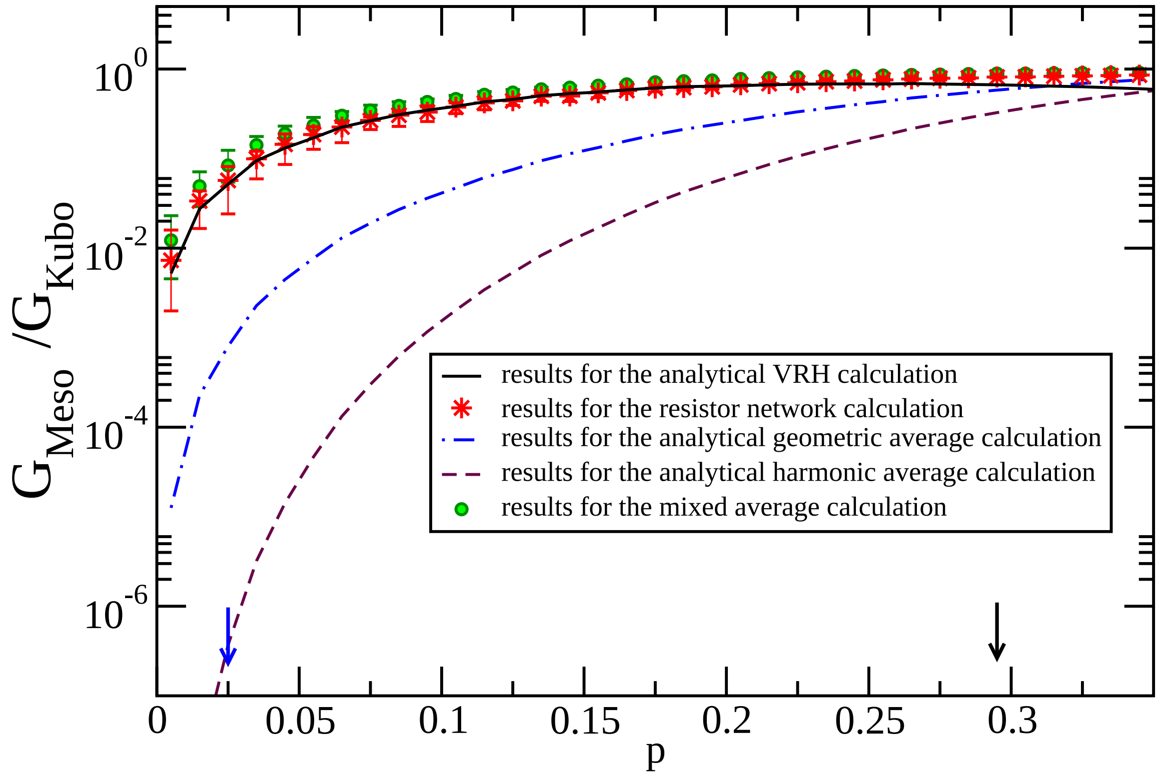
Localization:

$X \sim \mathrm{LogBox}$

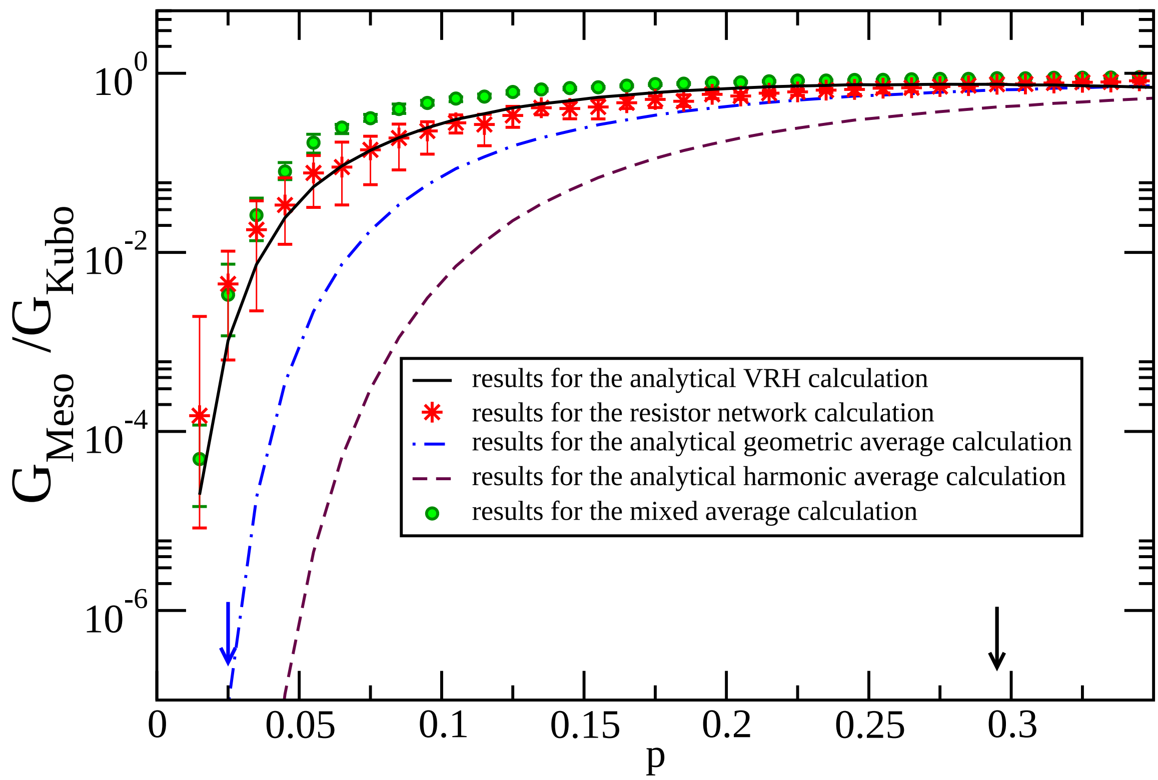
RMT based prediction for $G_{\mathbf{SLRT}}/G_{\mathbf{LRT}}$

RMT implied dependence on p

Log-normal distribution:



Log-box distribution:



The VRH estimate

$$\boldsymbol{G} \ = \ \pi\hbar \left(\frac{e}{L}\right)^2 \sum_{n,m} |v_{mn}|^2 \ \delta_T(E_n - E_F) \ \delta_\Gamma(E_m - E_n)$$

$$\boldsymbol{G} \ = \ \frac{1}{2}\left(\frac{e}{L}\right)^2 \varrho_{\mathrm{F}} \int \tilde{C}_{\mathrm{qm}}(\omega) \ \delta_\Gamma(\omega) \ d\omega$$

$$\tilde{C}_{\text{qm-LRT}}(\omega) \quad \equiv \quad 2\pi\varrho_{\mathrm{F}} \langle X \rangle$$

$$\tilde{C}_{\text{qm-SLRT}}(\omega) \quad \equiv \quad 2\pi\varrho_{\mathrm{F}} \overline{X}$$

where by definition: $\left(\frac{\omega}{\Delta}\right) \mathrm{Prob}\left(X > \overline{X}\right) \ \sim \ 1$

For strong disorder we get:

$$\overline{X} \ \approx \ v_{\mathrm{F}}^2 \exp\left(-\frac{\Delta_\ell}{\omega}\right)$$

$$\boldsymbol{G} \ \propto \ \int \exp\left(-\frac{\Delta_\ell}{|\omega|}\right) \exp\left(-\frac{|\omega|}{\omega_c}\right) d\omega$$

LRT, SLRT and beyond

$$-\dot{\Phi} \quad = \quad \text{electro motive force (RMS)}$$

$$G\,\dot{\Phi}^2 \quad = \quad \text{rate of energy absorption}$$

Semi linear response theory

[1] D. Cohen, T. Kottos and H. Schanz,
"Rate of energy absorption by a closed ballistic ring",
(JPA 2006)

[2] S. Bandopadhyay, Y. Etzioni and D. Cohen,
The conductance of a multi-mode ballistic ring,
(EPL 2006)

[3] M. Wilkinson, B. Mehlig, D. Cohen,
The absorption of metallic grains,
(EPL 2006)

[4] D. Cohen,
"From the Kubo formula to variable range hopping",
(PRB 2007)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos and D. Cohen,
The conductance of disordered rings,
(JPA / FTC 2008)

Beyond (semi) linear response theory

[6] D. Cohen and T. Kottos,
"Non-perturbative response of Driven Chaotic Mesoscopic Systems",
(PRL 2000)

[7] A. Stotland and D. Cohen,
"Diffractive energy spreading and its semiclassical limit",
(JPA 2006)

[8] A. Silva and V.E. Kravtsov,
Beyond FGR,
(PRB 2007)

[9] D.M. Basko, M.A. Skvortsov and V.E. Kravtsov,
Dynamical localization,
(PRL 2003)

Conclusions

(*) Wigner ($\sim$ 1955):
The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. Ballistic ring $\Longrightarrow$ log-normal distribution.
2. Strong localization $\Longrightarrow$ log-box distribution.
3. Resistors network calculation to get G_{SLRT}.
4. Generalization of the VRH estimate
5. SLRT is essential whenever the distribution of matrix elements is wide ("sparsity") or if the matrix has "texture".
6. Other applications of SLRT...