Black Hole Wave Packet

Average Area Entropy and Temperature Dependent (Doppler) Width

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Quantum gravity is still at large...

While a classically sharp event horizon is apparently mandatory for formulating (say) Schwarzschild black hole thermodynamics,

$$S = \frac{4\pi k_B G M^2}{\hbar c} \qquad T = \frac{\hbar c^3}{8\pi k_B G M}$$

Bekenstein entropy explodes *and* ***Hawking temperature vanishes*** *as* $\hbar \to 0$.

Once $\hbar$ *is switched on, the question* ***where is the horizon located?***

lacks an answer at the quantum or even at the semi-classical level.

It is not even clear whether the question is meaningful.

The ***quantum mechanical*** *Schwarzschild black hole is hereby described by a* ***non-singular minimal uncertainty wave packet*** *composed of plane wave eigenstates.*

The novel ingredients:

Average area entropy *and* ***Temperature dependent (Doppler) width.***

We carry out our analysis at the so-called ***mini super-spacetime*** *level without relying on theories beyond general relativity (string theory, the fuzzball proposal, or loop quantum gravity)*

A cosmological reminder:

$$ds^2 = -n^2(t)dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

Mini superspace: $-\frac{1}{16\pi G}\int (\mathcal{R} - 2\Lambda)\sqrt{-g}\, d^4x \longrightarrow \int \mathcal{L}(n, a, \dot{a})dt$

Up to a total derivative and an overall absorbable factor

$$\mathcal{L}(n, a, \dot{a}) = \frac{a}{n}\dot{a}^2 + na\left(\frac{1}{3}\Lambda a^2 - k\right)$$

leading to the classical field equation: $\boxed{\frac{\dot{a}^2}{n^2} + k = \frac{1}{3}\Lambda a^2}$

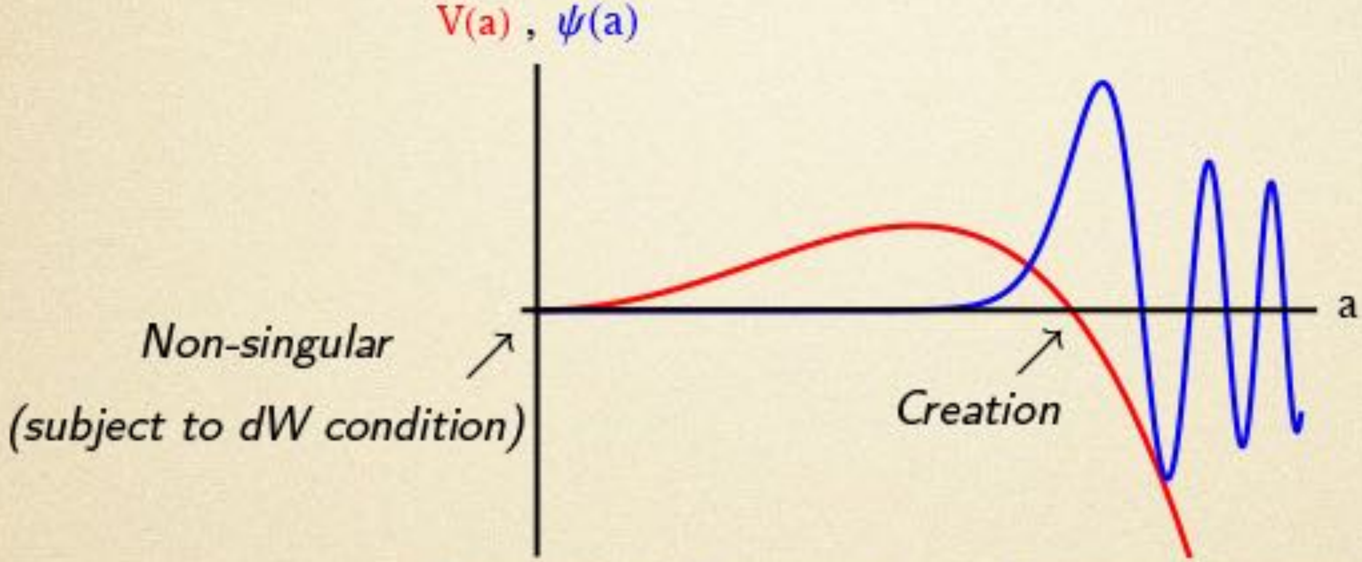
Note: *While* **post-fixing** *n(t)=1 is conventional, post-fixing a(t)=t is equally permissible even though it drives the FLRW equation* **algebraic**.

The momentum $p_a = \frac{2a\dot{a}}{n}$ *is accompanied by a primary constraint* $\phi = p_n \approx 0$

$$\mathcal{H} = \left(\frac{1}{4}p_a^2 + ka^2 - \frac{1}{3}\Lambda a^4\right)\frac{n}{a}$$

Require $\{\phi, \mathcal{H}\} = \frac{1}{a}\left(p_a^2 + 4ka^2 - \frac{1}{3}\Lambda a^4\right) = 0 \quad \Longrightarrow \quad \mathcal{H} = 0$

Wheeler-DeWitt Schrodinger equation $\mathcal{H}\psi(a) = 0$



Creation = Euclidean → Lorentzian transition

Various nucleation probability interpretations
Hartle-Hawking ['83], Vilenkin ['84], Linde ['84]

(i) Ironically, quantum cosmology is inherently ***static****.*

(ii) The classical solution plays no apparent role.

(iii) Spacetime is either strictly Lorentzian or else strictly Euclidean.

Pre-fixing (say) n(t)=1 is problematic $\mathcal{L}_1(a,\dot{a}) = a\dot{a}^2 + a\left(\frac{1}{3}\Lambda a^2 - k\right)$

introducing unphysical degrees of freedom $\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\Lambda}{3} + \frac{\xi}{a^3}$

However, pre-fixing a(t)=t is legitimate $\mathcal{L}_2(n,t) = \frac{t}{n} + nt\left(\frac{\Lambda}{3}t^2 - k\right)$

avoiding unphysical degrees of freedom via $n^2(t) = \left(\frac{1}{3}\Lambda t^2 - k\right)^{-1}$

The classical equations of motion residing from $\mathcal{H}_{2Total}$ serve as consistency checks for the two (a primary $p_n \approx 0$ and a secondary $\{p_n, \mathcal{H}_2\}_P = \frac{\partial \mathcal{L}_2}{\partial n} \approx 0$ second-class constraints.

The catch is, however, that $\{n, p_n\}_P = 1 \quad \Longrightarrow \quad \{n, p_n\}_D = 0$

The road to the standard quantization procedure has been blocked.

The missing ingredient is a generalized momentum, Dirac (rather than Poisson) conjugate to the lapse function.

A matter of measure? $\mathcal{S} = -\frac{1}{8\pi}\int (\mathcal{R} + 4\Lambda)\phi\sqrt{-g}\, d^4x$

The dilaton field, stripped from any Brans-Dicke kinetic term, is accompanied by a linear scalar potential (no $f(\mathcal{R})$ analogue)

*It is nevertheless a **dynamical** measure* $\Box\phi + V'_{eff}(\phi) = 0$

$$V(\phi) = 4\Lambda\phi \quad \Longrightarrow \quad V_{eff}(\phi) = \frac{1}{3}\int (\phi V' - 2V)d\phi = -\frac{2}{3}\Lambda\phi^2 + const$$

Constant Ricci scalar evolution (in the Jordan frame) is translated into the unified equation of state $\rho - 3P = 4\Lambda$

$$\rho = \Lambda + \frac{\xi}{a^4}\ , \quad P = -\Lambda + \frac{\xi}{3a^4}$$

Invoking the a(t)=t pre-gauging we arrive at

$$\mathcal{L}_2(n,t) \longrightarrow \mathcal{L}(n,\phi,\dot{\phi},t) = \frac{t}{n}(\phi + t\dot{\phi}) + nt\left(\frac{2}{3}\Lambda t^2 - k\right)\phi$$

$$\mathcal{L}_2(n,t) \longrightarrow \mathcal{L}(n,\phi,\dot{\phi},t) = \frac{t}{n}(\phi + t\dot{\phi}) + nt\left(\frac{2}{3}\Lambda t^2 - k\right)\phi$$

The classical solution

$$\boxed{n^2(t) = \left(\frac{1}{3}\Lambda t^2 - k + \frac{\xi}{t^2}\right)^{-1} \quad , \quad n(t)\phi(t) = const}$$

From $\psi(a)$ to $\psi(n,t)$ (not $\psi(n,\phi,t)$)

$$x(t) \equiv \frac{t^2}{n^2(t)} \;,\; y(t) \equiv \phi^2(t)t^2 \quad \Longrightarrow \quad \mathcal{H}(x,y,t) = -\sqrt{\frac{y}{x}}\,t\left(\frac{2}{3}\Lambda t^2 - k\right)$$

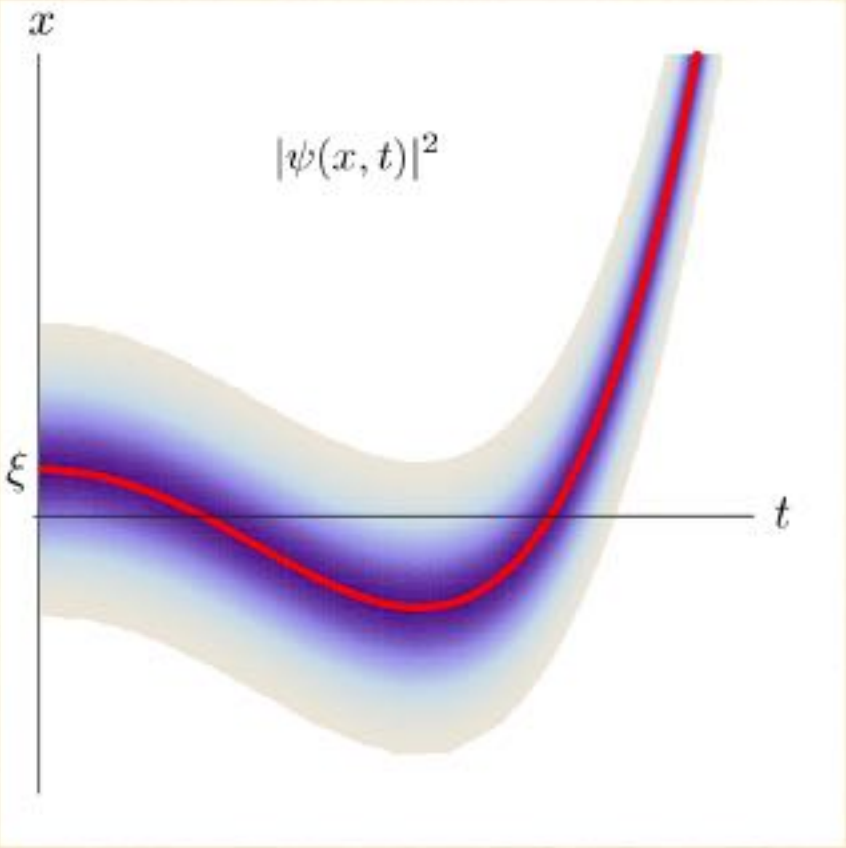
$$\left\{x, -\frac{1}{2}\sqrt{\frac{y}{x}}\right\}_D = 1 \quad \Longrightarrow \quad \sqrt{\frac{y}{x}} = 2i\frac{\partial}{\partial x}$$

The WdW equation $\quad -2it\left(\frac{2}{3}\Lambda t^2 - k\right)\frac{\partial\psi}{\partial x} = i\frac{\partial\psi}{\partial t}$

$$\psi(x,t) = \psi(x - \frac{1}{3}\Lambda t^4 + kt^2)$$

The quantum 'most classical' wave packet

$$\boxed{\psi(x,t) = \frac{e^{-\frac{1}{4\sigma^2}(x - x_{cl}(t))^2}}{(2\pi)^{\frac{1}{4}}\sqrt{\sigma}}}$$



Cosmological wave packet probability density

The most probable configuration (red) is the classical FLRW solution. For any time t, unlike in the Hartle-Hawking scheme, there is a probability that spacetime is Lorentzian ($x > 0$) and a complementary probability that it is Euclidean ($x < 0$).

The black hole connection

Combine the classical $n(t), \phi(t)$ *into the 5-dim KK line element*

$$ds_5^2 = -n^2(t)dt^2 + t^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) + \phi^2(t)dx_5^2$$

Use $t \to \rho, \;\; x_5 \to t, \; k = 1, \; \xi = m^2$

to recognize the Schwarzschild-deSitter Kaluza-Klein black hole.

This constitutes a justification to treat Black Holes on equal footing.

Denote the most general static spherically symmetric line element by

$$ds^2 = -\frac{y(r)}{2r}dt^2 + \frac{2r}{x(r)}dr^2 + r^2 d\Omega^2$$

Mini super spacetime: $-\frac{1}{16\pi G}\int \mathcal{R}\sqrt{-g}\, d^4x \longrightarrow \int \mathcal{L}(x, x', y, y', r)dr$

A gauge pre-fixing, namely defining r whose geometrical meaning is x,y-independent, has been harmlessly exercised.

We treat $\int \mathcal{L}(q, q', r)dr$ *in full* ***mathematical analogy*** *with* $\int \mathcal{L}(q, \dot{q}, t)dt$.

Technically, the t-evolution is traded for the r-evolution, both classically and quantum mechanically (York and Schmekel ['05]). To sharpen the point, our 'Hamiltonian' has nothing directly to do with the physical mass of the black hole.

Up to a total derivative and an overall absorbable factor

$$\mathcal{L}(x,x',y,y') = \left(\frac{3x'}{4} - 2\right)\sqrt{\frac{y}{x}} - \frac{y'}{4}\sqrt{\frac{x}{y}}$$

giving rise to two primary second class constraints

$$\phi_y = p_y + \frac{1}{4}\sqrt{\frac{x}{y}} \approx 0 \ , \ \ \phi_x = p_x - \frac{3}{4}\sqrt{\frac{y}{x}} \approx 0 \qquad \{\phi_y, \phi_x\} = \frac{1}{2\sqrt{xy}} \neq 0$$

Following **Dirac prescription**, *we are driven*
from the naive Hamiltonian $\mathcal{H} = p_x x' + p_y y' - \mathcal{L}$ *to the* **total Hamiltonian**

$$\mathcal{H}_T = 2\sqrt{\frac{y}{x}} + 2\frac{y}{x}\phi_y + 2\phi_x$$

Check: *The classical solution is (and is nothing but) the Schwarzschild solution*

$$\frac{y(r)}{2\omega^2 r} = \frac{x(r)}{2r} = 1 - \frac{2m}{r}$$

with no restrictions on the sign of the integration parameters m *and* ω.

Along the classical trajectories $\mathcal{H} = 2\omega$, *telling us that the 'Hamiltonian' is not the total physical mass of the system.*

To quantize the system it becomes crucial to first calculate the ***Dirac brackets***

$$\{A,B\}_D = \{A,B\}_+ \frac{\{A,\phi_y\}\{\phi_x,B\} - \{A,\phi_x\}\{\phi_y,B\}}{\{\phi_y,\phi_x\}}$$

just in case $\quad \frac{d*}{dr} = \{*,H_T\}_D + \left.\frac{\partial}{\partial r}\right|_D \qquad \left.\frac{\partial *}{\partial r}\right|_D \equiv \frac{\partial *}{\partial r} + \frac{\epsilon_{ij}}{\{\phi_1,\phi_2\}}\{*,\phi_i\}\frac{\partial \phi_j}{\partial r}$

$$\left\{p_x - \frac{3}{4}\sqrt{\frac{y}{x}},\ \star\right\}_D = \left\{p_y + \frac{1}{4}\sqrt{\frac{x}{y}},\ \star\right\}_D = 0$$

$\{x,y\}_P = 0 \quad$ *but* $\quad \{x,y\}_D = 2\sqrt{xy} \neq 0$

Counter intuitively, and potentially with far reaching consequences,

Two metric components do not Dirac commute !

$\boxed{\mathcal{H} = 2\sqrt{\frac{y}{x}}}$ $\quad$ ***Check:*** $\quad x' = \{x,\mathcal{H}\}_D = 2\ ,\quad y' = \{y,\mathcal{H}\}_D = \frac{2y}{x} \quad$ *ok*

The other metric component is represented by $\ y = \frac{1}{4}\mathcal{H}x\mathcal{H}$

$\{x, \frac{1}{2}\mathcal{H}\}_D = 1$ *paves the way for* $[x, \frac{1}{2}\mathcal{H}] = i\hbar$ *, hence* $\boxed{\mathcal{H} = -2i\hbar\frac{\partial}{\partial x}}$

The 2nd-class constraints $\phi_x\psi = \phi_y\psi \equiv 0$ *just define the operators* $p_{x,y}$ *.*

r-independent Schrodinger equation: $-2i\hbar\frac{\partial}{\partial x}\psi(x) = 2\omega\psi(x)$

The corresponding eigenstates are ***plane waves****.*

r-dependent Schrodinger equation: $-2i\hbar\frac{\partial}{\partial x}\psi(x,r) = i\hbar\frac{\partial}{\partial r}\psi(x,r)$

The full r-'evolution' is given by $\psi_\omega(x,r) = \frac{1}{\sqrt{4\pi}}e^{\frac{i}{\hbar}\omega(x-2r)}$

They are not localized and form a δ-normalizable set.

The most general solution is $\psi(x-2r)$ *.*

The 'most classical' $\Delta x\Delta\mathcal{H} = \hbar$ *wave packet*

$$\boxed{\psi(x,r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}}}$$

Two ***independent*** *parameters: m,σ*

Schwarzschild black hole wave packet

$$\boxed{\psi(x,r)=\frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}}}$$

in the range $-\infty<x<+\infty$ *, with* $\psi(\pm\infty,r)=0.$

Fourier transform $\tilde{\psi}(\mathcal{H})=\frac{2\sqrt{\sigma}}{(2\pi)^{\frac{1}{4}}}e^{-4\sigma^2\mathcal{H}^2}e^{2im\mathcal{H}}$

The Schwarzschild solution is the ***average*** *as well as the* ***most probable*** *configuration.* ***We expect this non-singular wave packet to capture the full semi-classical essence of black hole thermodynamics.***

In fact, one can construct an orthonormal tower of non-minimal uncertainty wave packets

$$\psi_n(x,r)=P_n(x-2r+4m)e^{-\frac{(x-2r+4m)^2}{64\sigma^2}} \qquad \Delta x\Delta\mathcal{H}=(2n+1)\hbar$$

none of which sharing however the Schwarzschild configuration as the most probable one.

For example, $\psi_1(x,r)\sim(x-2r+4m)e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}$

exhibits a ***bifurcated*** *most probable Schwarzschild configuration associated with masses* $m+\sigma$, $m-\sigma$.

To remind you, along the classical trajectories $\mathcal{H} = 2\omega$, telling us that the 'Hamiltonian' is not the total physical mass of the system.

Motivated by the classical Schwarzschild solution $m = \frac{1}{4}(2r - x_{cl}(r))$,

we identify the mass operator $\boxed{M(x,r) = \frac{1}{4}(2r - x)}$

$$\langle M \rangle = m \ , \ \langle M^2 \rangle = m^2 + \sigma^2$$

$\psi^\dagger \psi$ can then be translated into a **statistical mechanics mass spectrum**

$$\psi(x,r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}} \quad \Longrightarrow \quad \rho(M;m,\sigma) = \frac{e^{-\frac{(M-m)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

While $m \geq 0$ (the classical choice) is soon to be dictated on thermodynamical grounds, the M-distribution must cover, for the sake of quantum completeness, the entire range $-\infty < M < \infty$.

Who is afraid of negative masses? - Well, everybody...

The genuine mass of the quantum Schwarzscild black hole is m, however...

$$\psi(x,r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}} \quad \Longrightarrow \quad \rho(M;m,\sigma) = \frac{e^{-\frac{(M-m)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

At a certain location r, we can 'measure' various values for x. Associated with each particular value of x is a probability density $\rho(M;m,\sigma)$. M is here just a notation for $\frac{1}{4}(2r-x)$, motivated by the classical formula $m=\frac{1}{4}(2r-x_{cl}(r))$. It is only when we are tempted to give this quantum mechanical x the classical interpretation of $x(r)$, which generically differs from $x_{cl}(r)$, that one is driven to interpret M as an associated classical mass, which generically defers from m.

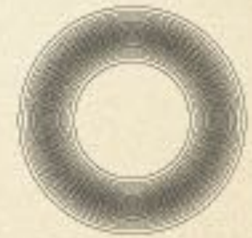
In analogy to the Erf-function tail probability to find a particle in a classically forbidden region (= negative kinetic term $p^2/2m$), the probability of having negative masses in the M-spectrum (for a non-negative m) is non-zero, and drops like $\sim e^{-\frac{m^2}{2\sigma^2}}$ towards the classical limit.

Where has the horizon gone?

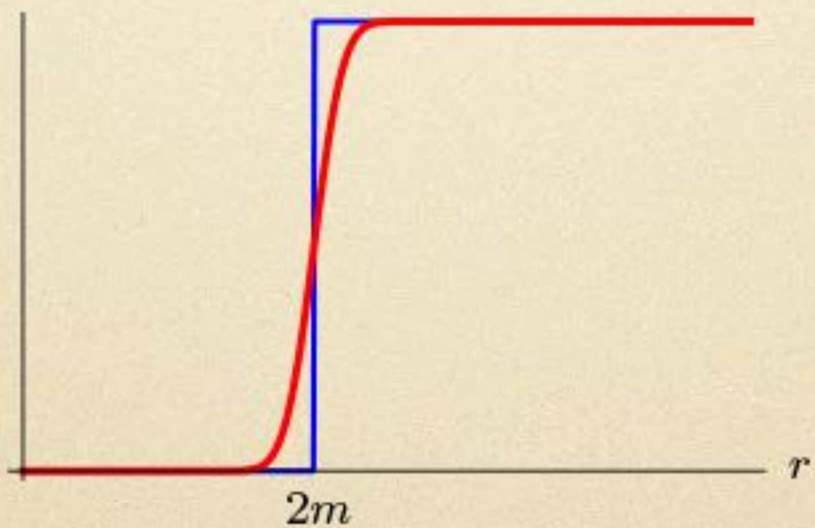
In fact, $\{x,y\}_D = 2\sqrt{xy} \neq 0$ clearly tells us that a sharp horizon is merely a classical gravitational concept.

However, in some sense one may still adopt the ***semi classical*** *interpretation of horizon fluctuations (Marolf ['05], York ['05]) or horizon profile (Casadio and Scardigli ['14]), with a probability density* $\rho(M; m, \sigma)$ *to find it at some radius M.*

Had we adopted the ***horizon profile*** *idea, it would have make sense to ask* ***where is the horizon actually located?*** *and consequently define an* ***information extract function***



$$I(r,m) = \int_{-\infty}^{\infty} \rho(M,m)\theta(r-2M)dM = \frac{1}{2}\left(1 + \mathrm{erf}(\frac{r-2m}{2\sqrt{2}\sigma})\right)$$



$$\psi(x,r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}} \quad \Longrightarrow \quad \rho(M;m,\sigma) = \frac{e^{-\frac{(M-m)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

There is nothing special going on near r=2m (and actually near r=0 as well)

$$\langle x\rangle = 2(r-2m) \to 0,\ \Delta x = 4\sigma$$
$$\langle y\rangle = \frac{r-2m}{32\sigma^2} \to 0,\ \Delta y \to \frac{k}{4\sigma}$$

apparent singularity removed

*This reminds us of the **fuzzball proposal** where the black hole arises from coarse graining over horizon-free non-singular geometries.*

Does an m=0 black hole wave packet make sense?

In principle it does, provided a finite width is permissible in such a case, giving rise to a fundamental quantum mechanical Schwarzschild black hole. In which case, negative/positive M are equally mandatory.

Treating the quantum mechanical black hole as a ***sub-system*** *[Bekenstein '73], its Gaussian* ***mass spectrum is temperature dependent****.*

$$\boxed{m = m(T),\ \sigma = \sigma(T)}$$

Following Fowler-Rushbrooke prescription ['39] to deal with such a sub-system

$$Z(\beta) = \sum_n \rho_n e^{-\beta F_n(\beta)}$$

The Boltzmann factor is traded for the Gibbs-Helmholtz factor

$$F + \beta \frac{\partial F}{\partial \beta} = E(\beta) \implies \beta F(\beta) = \int_{\beta_0}^{\beta} E(b) db$$

with β_0 *to be fixed on physical grounds.*

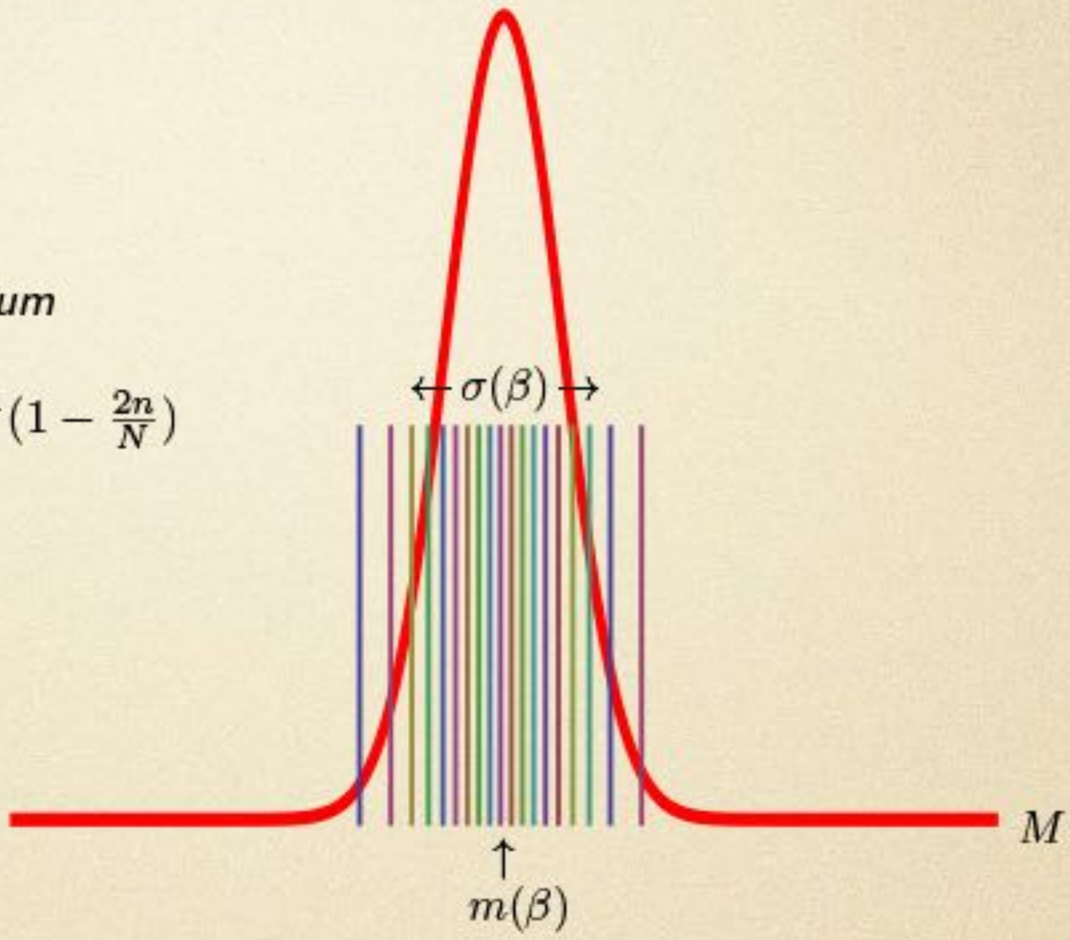
A pedagogical ***example****:* $E = const \Rightarrow \beta F \sim \beta\ ,\ \ E \sim \beta \Rightarrow \beta F \sim \frac{1}{2}\beta^2$

We find it convenient to discretize the problem by dividing the mass distribution into N equal probability and temperature independent sections.

$$\int_{M_n}^{M_{n+1}} \rho(M; m, \sigma) dM = \frac{1}{N}$$

Discretizing the mass spectrum

$$M_n(\beta) = m(\beta) - \sqrt{2}\sigma(\beta)\,\mathrm{erf}^{-1}(1 - \tfrac{2n}{N})$$



The mass distribution is divided into N equal probability sections, each of these wide sections is represented by a (temperature dependent) thin mass level.

System vs. sub-System

One level **system** $E = m$

$Z = e^{-\beta m} \qquad S = 0 \qquad$ *trivial*

Two level **system** $E = m \pm \sigma$

$Z = e^{-\beta m} \cosh \beta\sigma \qquad S = \log(\cosh \beta\sigma) - \beta\sigma \tanh \beta\sigma \qquad U = -\sigma \tanh \beta\sigma$

m irrelevant, $C = \dfrac{\beta^2 \sigma^2}{\cosh^2 \beta\sigma} > 0$

One level **sub-system** $E = m(\beta)$

☹ $Z = e^{-\beta m(\beta)} \qquad S = \beta^2 m'(\beta) \qquad U = m(\beta) + \beta m'(\beta)$

e.g. $m(\beta) \sim \beta \quad \Longrightarrow S \sim \beta^2,\ U \sim 2\beta$ *factor 2 discrepancy!*

☺ $Z = e^{-\int_0^\beta m(b)\, db} \quad S = \beta m(\beta) - \int_0^\beta m(b)\, db \quad U = m(\beta) \quad C = -\beta^2 m'(\beta)$

e.g. $m(\beta) \sim \beta \quad \Longrightarrow S \sim \frac{1}{2}\beta^2,\ U \sim \beta,\ C \sim -\beta^2 < 0$ Bekenstein-Hawking

The formal solution is $M_n(\beta) = m(\beta) - \sqrt{2}\sigma(\beta)\,\mathrm{erf}^{-1}(1 - \frac{2n}{N})$

Consequently, the Helmholtz free energy associated with the n-th mass level

$$\beta F_n = \int_{\beta_0}^{\beta} m(b)db - \sqrt{2}\,\mathrm{erf}^{-1}(1 - \frac{2n}{N}) \int_{\beta_0}^{\beta} \sigma(b)db$$

is now substituted into the partition function $Z(\beta) = \frac{1}{N}\sum_{n=1}^{N} e^{-\beta F_n(\beta)}$

Let $N \to \infty$ *and use* $\int_0^1 e^{\sqrt{2}\lambda\,\mathrm{erf}^{-1}(1-2\xi)}\, d\xi = e^{\frac{1}{2}\lambda^2}$ *to arrive at*

$$\boxed{Z(\beta) = e^{-\int_{\beta_0}^{\beta} m(b)\,db + \frac{1}{2}\left(\int_{\beta_0}^{\beta} \sigma(b)\,db\right)^2}}$$

The entropy $S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\log Z$ *is then the sum of two separate contributions*

$$\boxed{S(\beta) = S_m(\beta) + S_\sigma(\beta)}$$

The entropy

$$S(\beta) = S_m(\beta) + S_\sigma(\beta) \quad \textit{where} \quad \begin{aligned} S_m(\beta) &= \beta m(\beta) - \int_{\beta_0}^{\beta} m(b)db \\ S_\sigma(\beta) &= -\beta\sigma(\beta)\int_{\beta_0}^{\beta}\sigma(b)db + \frac{1}{2}\left(\int_{\beta_0}^{\beta}\sigma(b)db\right)^2 \end{aligned}$$

The internal energy

$$U(\beta) = m(\beta) - \sigma(\beta)\int_{\beta_0}^{\beta}\sigma(b)db$$

thereby closing on the 1st-law of thermodynamics $S'(\beta) = \beta U'(\beta)$

At this stage, $m(\beta),\ \sigma(\beta)$ *are two yet unspecified independent functions of* β.

The connection with black hole physics requires input beyond the mini super-spacetime model.

We thus adjust Bekenstein's area entropy ansatz $\boxed{S = \dfrac{\langle M^2\rangle}{2\eta^2} + c_S}$

by trading classical $\langle M\rangle^2 = m^2$ *for quantum mechanical* $\langle M^2\rangle = m^2 + \sigma^2$

$$S = \frac{\langle M^2 \rangle}{2\eta^2} + c_S$$

η *will be recognized as* $\sqrt{\dfrac{\hbar c}{8\pi G}}$ *as soon as the contact with Hawking temperature gets established.* c_S *is a constant to be determined.*

Having the 1st-law for a Gaussian mass distribution at our disposal, and recalling the ***compelling*** $m \leftrightarrow \sigma$ ***split****, the corresponding non-linear integral-differential equations to solve are*

$$\text{(i)} \qquad \beta m(\beta) - \int_{\beta_0}^{\beta} m(b)db = \frac{m^2(\beta)}{2\eta^2} + c_m$$

$$\text{(ii)} \qquad -\beta\sigma(\beta)\int_{\beta_0}^{\beta} \sigma(b)db + \frac{1}{2}\left(\int_{\beta_0}^{\beta} \sigma(b)db\right)^2 = \frac{\sigma^2(\beta)}{2\eta^2} + c_\sigma$$

The exact non-trivial solution of eq.(i) is noticeably β_0, c_m*-independent, namely*

$$\boxed{m(\beta) = \eta^2\beta} \qquad c_m = \frac{1}{2}\eta^2\beta_0^2$$

reassuring us that the reciprocal Hawking temperature is proportional, as expected (but non-trivial in the absence of a sharp horizon), to the ***necessarily positive average mass m>0****.*

The solution of eq.(ii) is a bit more complicated.

Define $f(\beta) \equiv \int_{\beta_0}^{\beta} \sigma(b)db$, *and attempt to solve* ***numerically.*** $\sigma(\beta) = f'(\beta)$

$$f'(\beta) = -\eta^2 \beta f(\beta) + \eta\sqrt{(1+\eta^2\beta^2)f^2(\beta) - 2c_\sigma} \quad \textit{subject to} \quad f(\beta_0) = 0$$

Before doing so, however, it is crucial to first fix β_0.

> *Fowler and Rushbrooke could not give a general rule for fixing* β_0*.They say: "The ambiguity has its counterpart in the use of the Gibbs Helmholtz equation to derive free energy from true energy. One needs to know, for instance, the entropy of the substance at some one particular temperature".*

Under $\beta_0 \to \beta_0 + \delta\beta_0$, $S(\beta)$ *gets shifted by a* β*-dependent amount.*

β_0 *is thus a physical parameter; its choice cannot be sensitive to* $S \to S + const$ *so its roots must be at the level of* $S'(\beta_0) = \beta_0 U'(\beta_0)$

The only ***tenable*** *choice is* $\boxed{\beta_0 = 0}$

It is furthermore a ***universal*** *choice in the sense that*

$$S(0) = S'(0) = U(0) = 0$$

The choice $\beta_0 = 0$ *suggests (but does not imply) that* $U'(0) = \eta^2 - \sigma_0^2$ *should vanish as well, in which case*

$$S(0) = S'(0) = U(0) = U'(0) = 0$$

There is yet a simpler argument to support the $\beta_0 = 0$ *choice. Hawking temperature tells us that choosing* β_0 *means choosing a special average mass, but there is no such a special mass.*

Fixing $\beta_0 = 0$ *also fixes the various constants floating around*

$$c_m = 0 \ , \quad c_\sigma = -\frac{\sigma_0^2}{2\eta^2} = c_S$$

that is $$\boxed{S = \frac{\langle M^2 \rangle}{2\eta^2} - \frac{\sigma_0^2}{2\eta^2}}$$

The equation $f'(\beta) = -\eta^2 \beta f(\beta) + \eta\sqrt{(1+\eta^2\beta^2)f^2(\beta) - 2c_\sigma}$ *tells us that*

$\sigma(\beta) = f'(\beta)$ *is a monotonically decreasing function of* β, *solely parameterized by the maximal width* σ_0.

for **large** $\eta\beta$: $\sigma(\beta) = \frac{s\sigma_0}{2\sqrt{\eta\beta}}\left(1 + \frac{1}{2s^2\eta\beta} + ...\right)$ $s \simeq 0.6185$

No log-terms at this stage.

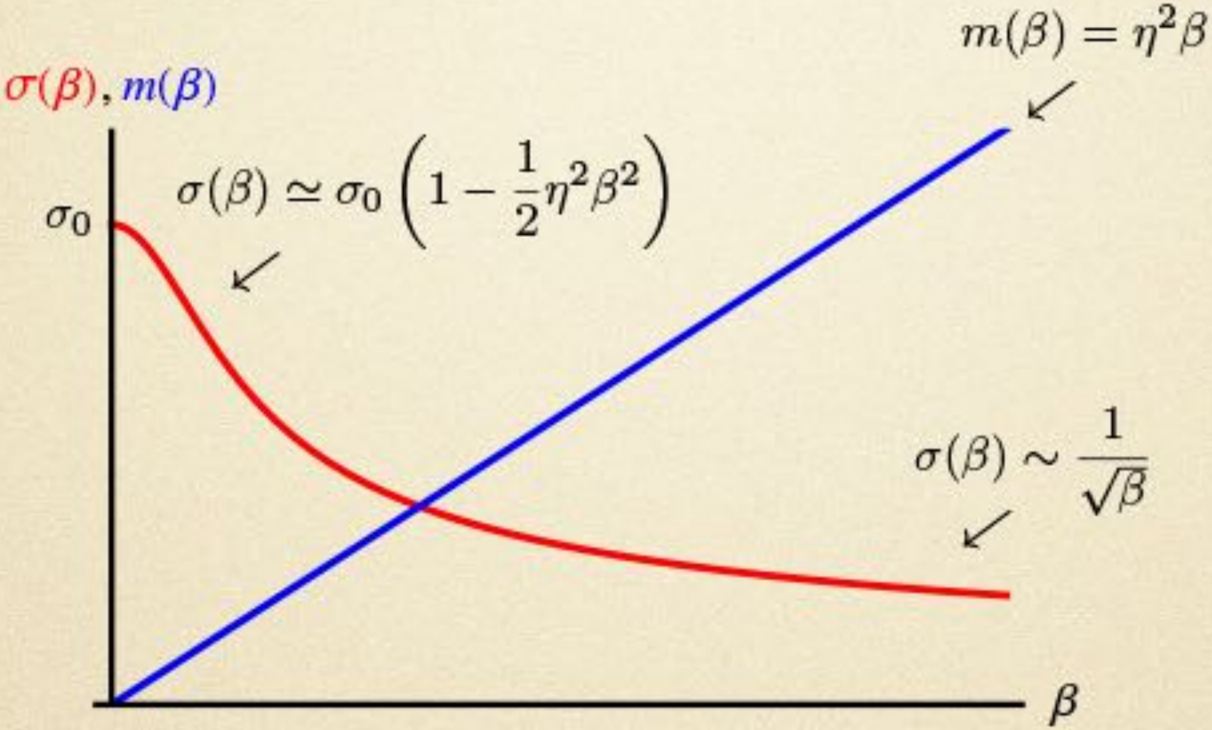
The Hawking temperature dependent width of the macro black hole wave packet highly reminds us (but apparently without any physics in common) of the Doppler broadening of spectral lines.

for **small** $\eta\beta$: $\sigma(\beta) = \sigma_0\left(1 - \frac{1}{2}\eta^2\beta^2 + \frac{3}{8}\eta^4\beta^4 + ...\right)$

Even the special case $\beta \to 0$, that is m=0, which classically leads to a flat spacetime, is quantum mechanically accompanied by a wave packet of non-vanishing width.

m and σ have been gradually elevated from being two independent parameters to two explicit functions of the Hawking temperature. Treating β as a parameter, one can now express $\sigma(m)$, and proceed to discuss the entropy $S(m)$ and the internal energy $U(m)$

Hawking temperature dependent wave packet



$$\langle M \rangle = m(\beta) \quad , \quad \langle M^2 \rangle = m^2(\beta) + \sigma^2(\beta)$$

$\langle M \rangle \to 0$, $M_{RMS} \to \sigma_0$ as $\beta \to 0$. σ_0 *is yet to be fixed.*

At the **classical limit** $m \gg \eta$ there are no surprises, with the leading Bekenstein-Hawking formulas acquire only tiny corrections

$$S(m) = \frac{m^2}{2\eta^2} - \frac{\sigma_0^2}{2\eta^2} + \frac{s^2\sigma_0^2}{8\eta m} + \dots \qquad U(m) = m - \frac{s^2\sigma_0^2}{2\eta} + \dots$$

At the **quantum regime** $m \leq \eta$ we find ourselves in an unfamiliar territory

$$S(m) = \left(1 - \frac{\sigma_0^2}{\eta^2}\right)\frac{m^2}{2\eta^2} + \frac{\sigma_0^2 m^4}{2\eta^6} + \dots \qquad U(m) = \left(1 - \frac{\sigma_0^2}{\eta^2}\right) m + \frac{2\sigma_0^2 m^3}{3\eta^4} + \dots$$

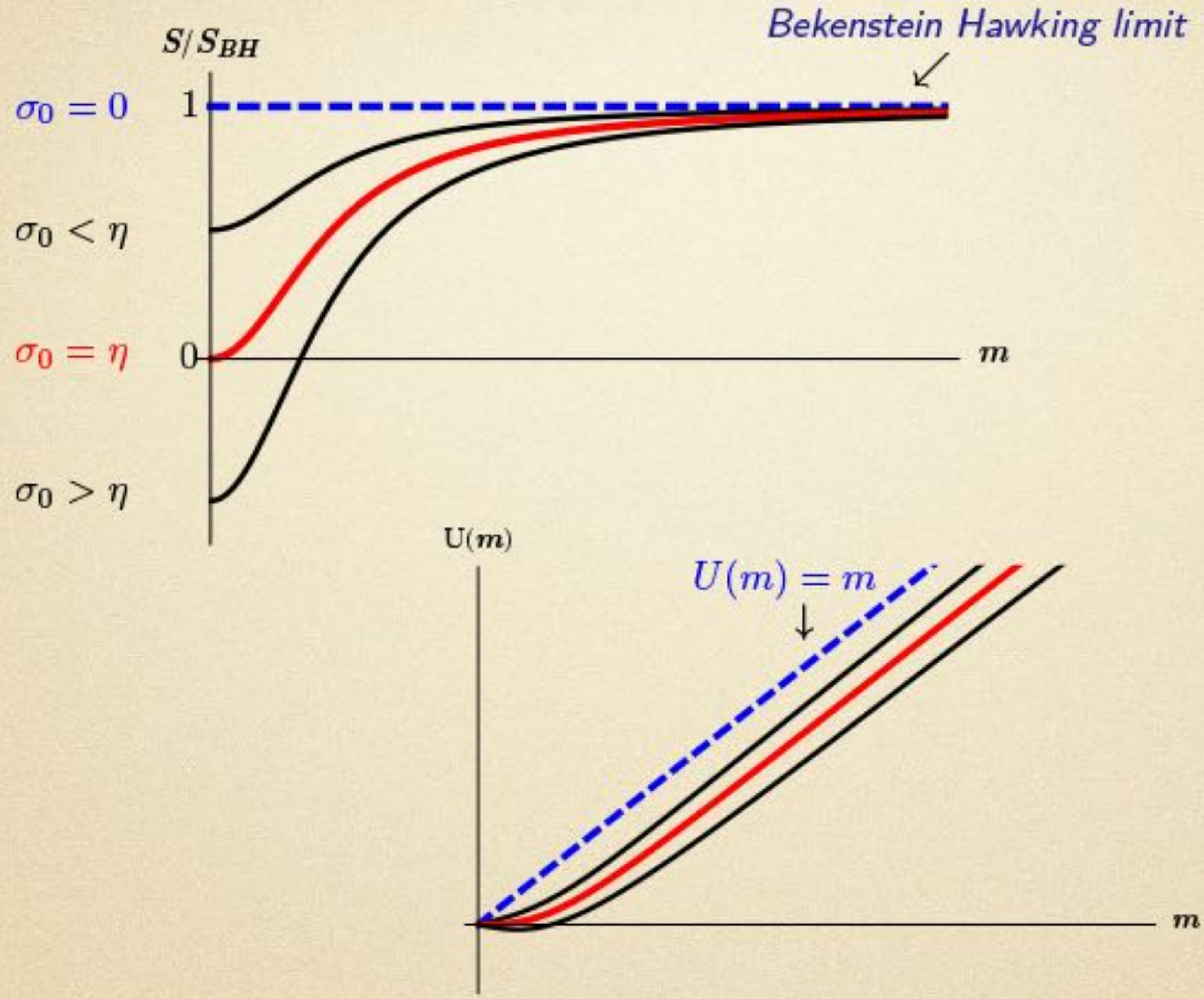
Regarding the value of σ_0, several possibilities arise:

(i) $\sigma_0 = 0$: Bekenstein-Hawking thermodynamics limit recovered.

(ii) $\sigma_0 > \eta$: The entropy develops a local maximum at m=0. The internal energy becomes negative in the neighborhood.

(iii) $\sigma_0 < \eta$: The entropy exhibits an absolute minimum at m=0, with $1 - \frac{\sigma_0^2}{\eta^2}$ serving as a suppression factor.

(iv) $\sigma_0 = \eta$: The entropy barely keeps its minimum at m=0, and the internal energy gives up its linear small-m behavior.


Bekenstein Hawking limit
S/S_BH
σ0 = 0
1
σ0 < η
σ0 = η
0
m
σ0 > η
U(m)
U(m) = m
m

Insist on attaching to the ***smallest*** $(m = 0,\ \sigma = \sigma_0)$ *quantum mechanical black hole wave packet a* ***minimal entropy****, but how to single out one particular value for* $\sigma_0 \leq \eta$ *?*

$$S(0) = S'(0) = U(0) = U'(0) = 0 \quad \Longrightarrow \quad \sigma_0 = \eta$$

Carrying zero entropy, this ***micro black hole*** *represents a single degree of freedom, and in this respect can be regarded elementary. It is characterized by a* ***finite root mean square mass*** $m_{RMS} = \eta$ *(consistent with the fact that Compton wavelength puts a limit on the minimum size of the region in which a mass can be localized), yet it is* ***divergently hot****, a feature which may play a crucial role at the final stage of black hole evaporation.*

At the ***classical limit*** $m \gg \eta$ *just a minor effect*

$$S(m) \simeq \frac{m^2}{2\eta} - \frac{1}{2} \qquad U(m) \simeq m - \frac{s^2\eta}{2}$$

At the ***quantum regime*** $m \leq \eta$ *a new ball game*

$$S(m) \simeq \frac{m^4}{2\eta^4} \qquad U(m) \simeq \frac{2m^3}{3\eta^2}$$

Quantum mechanical Schwarzschild black hole mass spectrum