Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

======= [Exercise 5811]

Mechanical model for symmetry breaking

An airtight piston of mass M is free to move inside a cylindrical tube of cross sectional area a. The tube is bent into a semicircular shape of radius R. On each side of the piston there is an ideal gas of N atoms at a temperature T. The angular position of the piston is φ (see figure). The gravitation field of Earth exerts a force Mg on the piston, while its effect on the gas particles can be neglected.
The partition function of the system can be written as $d\varphi$ integral over $\exp[-A(\varphi)]$. The variable φ is regarded as the "order parameter" of the system. A small difference ΔN in the occupation of the two sides is regarded as the conjugate field. The susceptibility is defined via the relation $\langle\varphi\rangle \approx \chi\Delta N$.

(1) Write an explicit expression for $A(\varphi)$.

(2) Find the coefficients in the expansion $A(\varphi) = (a/2)\varphi^2 + (u/4)\varphi^4 - h\varphi$.

(3) Deduce what is the critical temperature T_c.

(4) Using Gaussian approximation find what is χ for $T > T_c$.

(5) Using Gaussian approximation find what is χ for $T < T_c$.

(6) Sketch a plot of χ versus T indicating by dashed lines the Gaussian approximations and by solid line the expected exact result. Write what is the range ΔT around T_c where the Gaussian approximation fails.

(7) What is the way to take the "thermodynamic limit" such as to have a phase transition at finite temperature?

(8) In reality, as the temperature is lowered, droplets condense on the walls of the left (larger) chamber. What do you expect to find in the right chamber (gas? liquid? both?).

Guidelines: In items (4) and (5) simplify the result assuming $T \sim T_c$ and express it in terms of T_c and $T - T_c$. The final answer should include one term only. Care about numerical prefactors - their correctness indicates that the algebra is done properly. In item (7) you are requested to identify the parameter that should be taken to infinity in order to get a "phase transition". Please specify what are the other parameters that should be kept constant while taking this limit.