E5010: One dimensional hard sphere gas

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The problem:

N beads of diameter a and mass m are threaded over a wire of length L. Assume $N >> 1$, $Na << L$. The system is in a thermal equilibrium and its temperature is T.

(1) Find the force F that is acting on the edges of the wire. Write the result using the form $F = N\frac{T}{L_{eff}}$. Express L_{eff} using the variables T, m, N, a.
(2) Explain the physical meaning of the result.

Hints:
(a) When calculating the partition function, notice that if permutations were allowed it would have caused $Z \to N!Z$.
(b) Assume that a typical distance between two beads is much larger than a.
(c) To calculate the product $A = \prod_{n=1}^{N} a_n$ look at the sum $\ln(A)$ and use reasonable approximations to calculate it.

The solution:

(1) We begin by calculating the partition function. We denote by $x_1 ... x_N$ the positions of the beads on the wire where x_1 is the position of the bead closest to the edge of the wire at $x = L$. The partition function is:

$$Z = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \mathrm{d}p_N e^{-\beta\frac{p_N^2}{2m}} \cdot ... \cdot \int_{-\infty}^{\infty} \mathrm{d}p_1 e^{-\beta\frac{p_1^2}{2m}} \cdot \int_{0}^{L-(N-1)a} \mathrm{d}x_N \cdot ... \cdot \int_{x_3+a}^{L-a} \mathrm{d}x_2 \cdot \int_{x_2+a}^{L} \mathrm{d}x_1$$

Where in this calculation we neglect edge effect. We calculate the momentum integrals and get:

$$Z = (\frac{1}{\lambda_T})^N \cdot \int_{0}^{L-(N-1)a} \mathrm{d}x_N \cdot ... \cdot \int_{x_3+a}^{L-a} \mathrm{d}x_2 \cdot \int_{x_2+a}^{L} \mathrm{d}x_1$$

We denote by I_n the product: $\int_{0}^{L-(n-1)a} \mathrm{d}x_n \cdot ... \cdot \int_{x_2+a}^{L} \mathrm{d}x_1$

We calculate I_2:

$$I_2 = \int_{x_3+a}^{L-a} \mathrm{d}x_2 \cdot \int_{x_2+a}^{L} \mathrm{d}x_1 = \int_{x_3+a}^{L-a} \mathrm{d}x_2 \cdot (L - x_2 - a)$$

We change the variable of integration to be $y \equiv L - x_2 - a$ and get:

$$I_2 = \int_{L-x_3-2a}^{0} (-y)\mathrm{d}y = \frac{1}{2}(L - (x_3 + 2a))^2$$

Now for I_n we assume:

$$I_n = \int_{x_{n+1}+a}^{L-(n-1)a} \mathrm{d}x_n I_{n-1} = \frac{(L-(x_{n+1}+na))^n}{n!}$$

and show that if the formula for I_n is true then the formula is true for I_{n+1}:

$$I_{n+1} = \int_{x_{n+2}+a}^{L-na} \mathrm{d}x_{n+1} I_n = \frac{1}{n!} \int_{x_{n+2}+a}^{L-na} \mathrm{d}x_{n+1} (L-(x_{n+1}+na))^n = \frac{(L-(x_{n+2}+(n+1)a))^{n+1}}{(n+1)!}$$

Hence, by induction the formula for I_n is true. Now for N beads, the position of the bead $N+1$ is the position of the edge of the wire ($x=0$) minus the diameter of the bead. So we get:

$$I_N = \frac{(L-(N-1)a)^N}{N!}$$

So the partition function is:

$$Z = (\frac{1}{\lambda_T})^N \cdot \frac{(L-(N-1)a)^N}{N!}$$

We now calculate $\ln Z$:

$$\ln Z = \ln[(\frac{1}{\lambda_T})^N \frac{1}{N!}] + N \ln(L-(N-1)a)$$

For $L >> Na$ and for $N >> 1$ we get:

$$\ln Z \simeq \ln[(\frac{1}{\lambda_T})^N \frac{1}{N!}] + N \ln L - \frac{N^2 a}{L}$$

Now we calculate the force that is acting on the edge of the wire:

$$F = T \frac{\partial \ln Z}{\partial L} = \frac{NT}{L^2/(L+Na)}$$

For $L >> Na$ we can write: $\frac{L^2}{L+Na} \simeq L - Na$ and the expression for the force becomes:

$$F \simeq \frac{NT}{L-Na} = \frac{NT}{L_{eff}}$$

(2) The result we have got is exactly the result of the kinetic picture. In the kinetic picture, in 1D, the force is given by $F = \frac{Nmv^2}{L}$ and for $mv^2 = T$ it becomes $F = \frac{NT}{L}$. This is not surprising because all of the assumptions we made here are corresponding to the kinetic theory postulates. The expression for L_{eff} is the part of the wire that is not occupied by the beads.