A25 (2009 3.1)
(exam 2008A.1)

$$\mathcal{H} = \frac{p^2}{2m} \pm \mu B$$

a) $$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$$

$$Z = Z_{id} \times Z_{spin} = \underbrace{\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N}_{\text{ideal gas}} \cdot \underbrace{\left(2\cosh(\beta\mu B)\right)^N}_{\text{spin system}}$$

$\downarrow$

$$F = F_{kin} + F_{spin}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = S_{kin} + N\frac{\partial}{\partial T}\left(k_B T \ln(2\cosh(\beta\mu B))\right)$$

with $$S_{kin} = N k_B \ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2}k_B$$

$$S_{spin} = N k_B \ln(2\cosh(\beta\mu B)) - \beta\mu B \tanh(\beta\mu B)$$

$$\lim_{x\to\infty} \ln(2\cosh x) = \lim_{x\to\infty} \ln(e^x + e^{-x}) = x$$

$$\lim_{x\to\infty} \tanh(x) = 1 \rightarrow \lim_{x\to\infty} x\tanh(x) = x$$

לכן עבור $B\to\infty$ $S_{spin}\to 0$ כצפוי.

ועבור $B\to 0$ $S_{spin} = N k_B \ln 2$

$$S = S_{kin} + S_{spin} = \text{const}$$

תהליך אדיאבטי:

$$\rightarrow S_{kin}(T_i) + S_{spin}(B, T_i) = S_{kin}(T_f) + S_{spin}(0, T_f)$$

$$\rightarrow \boxed{k_B N \ln\left(\frac{\lambda^3_{T_f}}{\lambda^3_{T_i}}\right) = \frac{3}{2}k_B N \ln\frac{T_i}{T_f} = S_{spin}(0, T_f) - S_{spin}(B, T_i)}$$

b) As $B\to\infty$ $S_{spin}(T_i, B) = 0$

$$\rightarrow \ln\frac{T_f}{T_i} = -\frac{2}{3}\ln 2 \rightarrow \boxed{T_f = \frac{T_i}{2^{2/3}}}$$

c) d dimensions: $\lambda^3 \to \lambda^d$, spin has $2S+1$ states

$$\frac{d}{2}\ln\left(\frac{T_f}{T_i}\right) = -\ln(2S+1) \rightarrow T_f = \frac{T_i}{(2S+1)^{2/d}}$$

(HW 2008 7.1) (2009 3.2)

A17 N hard spheres, single sphere vol w, total Volume V only h.c. interactions. $a = V^{\frac{1}{3}}$

a) $$Z = \frac{1}{N!}\frac{1}{(2\pi\hbar)^{3N}} \underbrace{\int_{-\infty}^{\infty} \prod_i d\bar{p}_i \, e^{-\frac{\beta \bar{p}_i^2}{2m}}} \underbrace{\int_{\forall_{ij} |X_i - X_j| > a} \prod_i d\bar{x}_i}$$

$$\approx \frac{1}{N!} \lambda_T^{3N} V(V-w)(V-2w)\dots(V-(N-1)w)$$

validity of approx: the physical meaning: 1st ball has available volume V, 2nd ball available volume $(V-w)$ ... N^{th} ball has available vol $(V-(N-1)w)$.

Ignores excluded volume between neighbours - "packing" which is a 3-body effect.

b) $F = -k_B T \ln Z$

Notice that $V(V-w)\dots(V-(N-1)w) = V\cdot[(V-w)(V-(N-1)w)]\cdot[(V-2w)(V-(N-2)w)]$

Giving $$Z \approx \frac{\lambda_T^{3N}}{N!}\left(V - \frac{Nw}{2}\right)^{N-1} V = \frac{1}{N!}\underbrace{(\lambda_T^3 V)^N}_{\text{free gas}} \underbrace{\left(1 - \frac{Nw}{2V}\right)^{N-1}}_{\text{h.c. interaction}}$$

$$F = F_{free} - k_B T (N-1)\ln\left(1 - \frac{Nw}{2V}\right)$$

$$S = -\frac{\partial F}{\partial T}\Big|_{V,N} = S_{free} + k_B (N-1)\ln\left(1 - \frac{Nw}{2V}\right)$$

The approximation used is valid when $(V-aw)(V-(N-a)w) \approx \left(V - \frac{Nw}{2}\right)^2$ which can be expanded such that

$$V^2 - VNw + w^2 a(N-a) \approx V^2 - VNw + \frac{N^2w^2}{4} \rightarrow a(N-a) \approx \frac{N^2}{4} \rightarrow N \approx 2a$$

c) Equation of state:

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = -\left(\frac{\partial F_{free}}{\partial V}\right)_{N,T} + k_B T(N-1) \cdot \frac{\frac{Nw}{2V^2}}{1 - \frac{Nw}{2V}}$$

$$\approx \frac{k_B T N}{V} + \frac{k_B T N}{V}\left(\frac{1}{\frac{2V}{Nw} - 1}\right)$$

$$PV = k_B T N\left(1 + \frac{1}{\frac{2V}{Nw} - 1}\right) = k_B T N\left(\frac{2V/Nw}{\frac{2V}{Nw} - 1}\right)$$

$$PV = k_B T N \cdot \frac{1}{1 - \frac{Nw}{2V}} \longleftrightarrow P\left(V - \frac{Nw}{2}\right) \approx N k_B T$$

for $V \gg Nw$ we have $PV \approx k_B T N$

for $V \approx Nw$ we have $PV \approx \frac{1}{2} k_B T N$

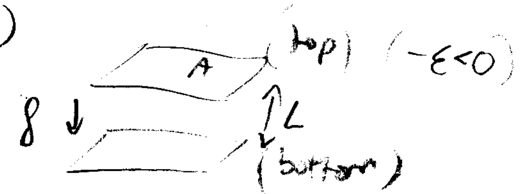
d) Isothermal compressibility: $-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}$

we have $\left(V - \frac{Nw}{2}\right) \approx \frac{N k_B T}{P}$

$$-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{-1}{V}\left(\frac{\partial\left(V - \frac{Nw}{2}\right)}{\partial P}\right)_{T,N} = \frac{N k_B T}{V P^2} > 0$$

as required.

$\boxed{A1B}$ (2009 3.3)



a) $\mathcal{H} = \frac{p^2}{2m} + gz$, $Z_1(z) = \left(\frac{1}{2\pi\hbar}\right)^3 \int_{-\infty}^{\infty} d^3p\, e^{-\beta\frac{p^2}{2m}} \int_{z-\frac{\Delta z}{2}}^{z+\frac{\Delta z}{2}} e^{-\beta g z}\, dz \cdot A$

$\rightarrow Z \approx \frac{1}{N(z)!}\left(\frac{A\Delta z}{\lambda^3} e^{-\beta g z}\right)^{N(z)}$

$\frac{F(z)}{N(z)} = +kT\ln(n(z)\lambda^3) + gz - kT$.

$\mu_{3D}(z) = \frac{\partial F(z)}{\partial N(z)} = gz + kT\ln(n(z)\lambda^3) = \mu_{id} + gz$

In equilibrium $\mu(z) = \mu = \mu(z=0)$

$n(z) = \frac{1}{\lambda^3} e^{\beta(\mu - gz)} = n(0) e^{-\beta g z}$

$n_0 = \frac{1}{L}\int_0^L n(z)\,dz = n(0)\cdot \frac{1-e^{-\beta g L}}{\beta g L}$ $\quad$ $T \gg \varepsilon$

$\downarrow$

$$\boxed{n(z) = n_0 \frac{\beta g L}{1-e^{-\beta g L}} e^{-\beta g z}}$$

b) $Z_{plate} = \begin{cases} \frac{1}{N!}\left(\frac{A}{\lambda^2} e^{\beta\varepsilon - \beta g L}\right)^{N(L)} & \text{top plate} \\ \frac{1}{N!}\left(\frac{A}{\lambda^2} e^{\beta\varepsilon}\right)^{N(0)} & \text{bottom plate} \end{cases}$

Similarly for $(z=0,L)$ $\quad F_{2D} = \left[-\varepsilon + gz + kT\ln(n_{2D}(z)\lambda^2) + kT\right] N(z)$

$\rightarrow \mu_{2D} = -\varepsilon + gz + kT\ln(n_{2D}\lambda^2) = \mu_{3D} = kT\ln(n(0)\lambda^3)$

$\downarrow$

$n_{2D} = n(0)\lambda\, e^{\beta(\varepsilon - gz)} = n_0 \lambda e^{\beta\varepsilon} \frac{\beta g L}{1-e^{-\beta g L}} \cdot \begin{cases} e^{-\beta g L} & \text{top plate} \\ 1 & \text{bottom plate} \end{cases}$

c) Force:

$$P(0)\cdot A = n(0)kT\cdot A \qquad \text{(ideal gas)}$$

$$P(L)A = n(L)\cdot kTA$$

$$\left[P(0)-P(L)\right]\cdot A = n_0\cdot\frac{\beta g L}{1-e^{-\beta g L}}\left(1-e^{-\beta g L}\right)\cdot kT\cdot A$$

$$= n_0 g L A$$

הכח כלפי מעלה כי $(n(0)>n(L))$

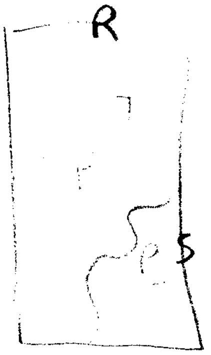
על החלקיק פועל כח g לכן סה״כ הכח של החלקיקים

$$F_{part} = g\, n_0 L A$$

כלפי מעלה

ולכן סה״כ הכוחות $= 0$.

A09 . 2003 3-1) + (2003 3-4)



(c)

עבור מצב מיקרוסקופי r של המערכת:

$$\ln \Omega_R(E_0-E_r, V_0-V_r) = \ln \Omega_R(E_0,V_0) - \underbrace{\frac{\partial \ln \Omega_R}{\partial E}\Big|_{E_0}}_{=\beta} E_r - \underbrace{\frac{\partial \ln \Omega_R}{\partial V}\Big|_{V_0}}_{=\beta P} V_r$$

$$P_r = \frac{e^{-\beta(E_r + PV_r)}}{Z(P,T)}$$ במצב r

[נסכם במצב קבוע לצד הקנוני $P_r \sim e^{-\beta E_r}$

קנוני $F = -kT\ln Z = U - TS$

כאן נקבל $-kT\ln Z = U + PV - TS = G$]

$$Z_N(P,T) = \sum_r e^{-\beta E_r - \beta P V_r} = \sum_V e^{-\beta PV} \sum_{r \text{ בנפח } V} e^{-\beta E_r}$$

$$= \sum_V e^{-\beta PV - \beta F(V)}$$

$$\frac{\partial}{\partial V}\left(PV + F\right)_{P,T} = P + \frac{\partial F}{\partial V}\Big|_{P,T,\langle V\rangle} = 0$$ הביטוי המינימלי סביב

$$Z = e^{-\beta(P\langle V\rangle + F(\langle V\rangle))}\, e^{-\beta \frac{\partial^2 F}{\partial V^2}(V-\bar{V})^2}$$ ולכן נקבל

(תרומה כזו [illegible])

$$-kT\ln Z = P\langle V\rangle + F + O(\ln N) = G(P,T,N)$$

$\boxed{A10}$ (2009 3.4)

משתמש בכל ה- $\boxed{A09}$ ונקבל

$$Z_1(f,T) = \int d\Omega\, e^{-\beta f a \cos\theta} = 2\pi \int_{-1}^{1} d(\cos\theta)\, e^{-\beta f a \cos\theta}$$

$$= \frac{4\pi}{\beta f a} \sinh(\beta f a)$$

$$Z_N(f,T) = Z_1^N = \left(\frac{4\pi}{\beta f a}\right)^N \sinh^N(\beta f a)$$

$$\langle L \rangle = -\left(\frac{\partial G}{\partial f}\right)_{T,N} = +kT \frac{\partial \ln Z}{\partial f}\Big|_{T,N} = -kTN \frac{\partial}{\partial f}\left(-\ln f + \ln\sinh(\beta f a)\right)$$

הסימן הפוך כי מתיחה זה כיוון שעושה עבודה על המערכת ולא להפך

$$= +kTN\left(\frac{-1}{f} + \frac{\cosh(\beta f a)\cdot \beta a}{\sinh(\beta f a)}\right)$$

$$= \frac{aN}{\beta f a}\left(\coth(\beta f a)\cdot \beta f a - 1\right)$$

$x = \beta f a$ נסמן

$$\langle L \rangle = aN\left(\coth(x) - \frac{1}{x}\right)$$