1.

We set the k_F vector from the requirement that the number of states equals the number of electrons:

$$2\frac{\pi k_F^2 S}{(2\pi)^2} = N \tag{1}$$

where the 2 factor is for spin degeneracy. The result is

$$k_F = \sqrt{2\pi n} \tag{2}$$

2.

The number of states per unit area up to wave number k is the area of the k-space circle divided by the k-space area of a single states,

$$\mathcal{N}(k) = 2\frac{\pi k^2}{(2\pi)^2} \tag{3}$$

and using the relation $\mathcal{E} = \hbar^2 k^2/2m$ we get

$$\mathcal{N}(\mathcal{E}) = 2\frac{m\mathcal{E}}{2\pi\hbar^2} \tag{4}$$

from which we derive the density of states

$$g(\mathcal{E}) = \frac{d\mathcal{N}}{d\mathcal{E}} = \frac{m}{\pi\hbar^2} \tag{5}$$

3.

The pressure in 2D is given by

$$P = -\left(\frac{\partial F}{\partial S}\right)_N \underset{T=0}{=} -\left(\frac{\partial E}{\partial S}\right)_N \tag{6}$$

At $T = 0$ the energy is

$$E = S\int_0^{\mathcal{E}_F} g(\mathcal{E})\mathcal{E}d\mathcal{E} = S\int_0^{\mathcal{E}_F} \mathcal{E}\frac{m}{\pi\hbar^2}d\mathcal{E} = S\frac{m}{2\pi\hbar^2}{\mathcal{E}_F}^2 = \frac{\pi\hbar^2 N^2}{2mS} \tag{7}$$

so the pressure is,

$$P = \frac{\pi\hbar^2}{2m}\left(\frac{N}{S}\right)^2 \tag{8}$$

4.

From

$$n = \int_{-\infty}^{\infty} d\mathcal{E} g(\mathcal{E}) n_F(\mathcal{E}) \tag{9}$$

we have

$$n = \frac{m}{\pi\hbar^2}\int_0^{\infty} d\mathcal{E}\frac{1}{e^{(\mathcal{E}-\mu)/k_B T}+1} \tag{10}$$

By making the substitution $x = e^{(\mathcal{E}-\mu)/k_B T}$ and $d\mathcal{E} = k_B T dx/x$ the integral becomes

$$n = \frac{m k_B T}{\pi \hbar^2} \int_{e^{-\mu/k_B T}}^{\infty} \frac{dx}{x(x+1)} = \frac{m k_B T}{\pi \hbar^2} \ln \left(e^{\mu/k_B T} + 1 \right) \tag{11}$$

and solving for μ we obtain

$$\mu = k_B T \ln \left(e^{\pi n \hbar^2 / m k_B T} - 1 \right) \tag{12}$$

1.

The force equation is

$$M\ddot{u}_n = G\left(u_{n+1} + u_{n-1} - 2u_n\right) \tag{1}$$

and its solution is given by

$$u_n \sim e^{i(kna-\omega t)} \tag{2}$$

which gives the dispersion relation

$$-M\omega^2 = -G\left(2 - e^{ika} - e^{-ika}\right)$$

$$\omega^2 = \frac{4G}{M}\sin^2\left(\frac{ka}{2}\right) \tag{3}$$

2.

$$k = k_R + ik_I$$

$$\omega = \sqrt{\frac{4G}{M}}\sin\left(\frac{a}{2}(k_R + ik_I)\right) = \sqrt{\frac{4G}{M}}\left(\frac{e^{ik_Ra/2}e^{-k_Ia/2} - e^{-ik_Ra/2}e^{k_Ia/2}}{2i}\right) \tag{4}$$

We demand $\mathrm{Im}[\omega] = 0$ to find oscillatory solutions

$$\cos\left(k_R\frac{a}{2}\right)e^{-k_I\frac{a}{2}} - \cos\left(k_R\frac{a}{2}\right)e^{k_I\frac{a}{2}} = 0$$

$$\cos\left(k_R\frac{a}{2}\right)\sinh\left(k_I\frac{a}{2}\right) = 0$$

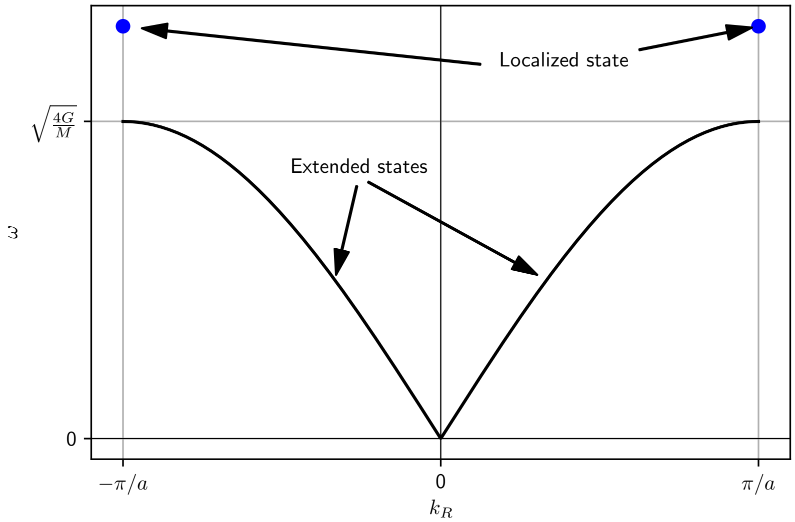
$k_I = 0$ gives the bulk solution found in section 1. The localized solutions have $k_I \neq 0$ so

$$\frac{ak_R}{2} = \pm\frac{\pi}{2} \implies k_R = \pm\frac{\pi}{a}$$

so the solution is

$$u_n \sim e^{i(k_R+ik_I)na}e^{-i\omega t} = (-1)^n e^{-k_Ina}e^{-i\omega t} \tag{5}$$

3.



4.

The equation for the first mass, together with the equation for the other masses are

$$\begin{cases} m\ddot{u}_0 = G\left(u_1 - u_0\right) \\ M\ddot{u}_n = G\left(u_{n+1} + u_{n-1} - 2u_n\right) \quad , n > 0 \end{cases}$$

Substituting the solution found in section 2,

$$\begin{cases} \omega^2 m = G\left(1 + e^{-k_I a}\right) \\ \omega^2 M = G\left(2 + e^{k_I a} + e^{-k_I a}\right) = G\left(1 + e^{-k_I a}\right)\left(1 + e^{k_I a}\right) \end{cases} \tag{6}$$

Dividing the equation one obtains

$$\frac{M}{m} = e^{k_I a} + 1 \implies k_I = \frac{1}{a}\ln\left(\frac{M}{m} - 1\right) \tag{7}$$

We are given that $M/m > 2$, otherwise we would get $k_I < 0$ which would make the solution diverge at $n \to \infty$. For ω we look back at the dispersion relation so

$$\omega^2 = \frac{G}{m}\left(1 + \frac{1}{e^{k_I a}}\right) = \frac{G}{m}\left(\frac{M}{M - m}\right) \tag{8}$$

1.

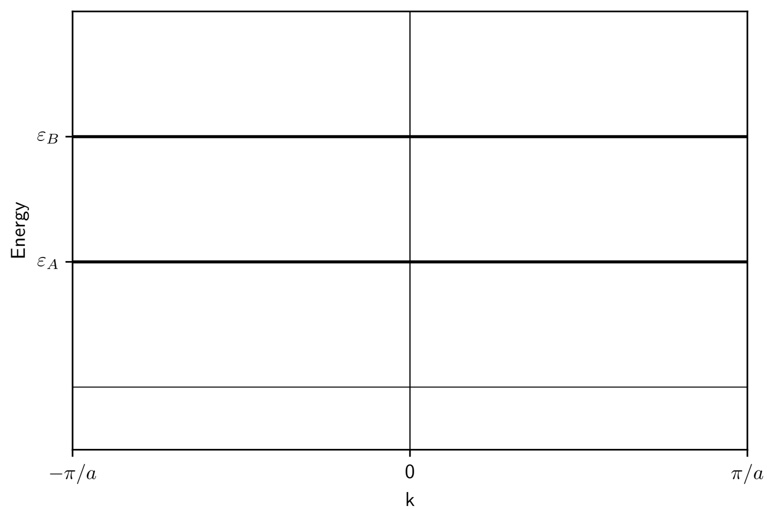
At $t = 0$ obviously $\langle \phi_k | \hat{H} | \psi_{k'} \rangle = 0$, so we only need to check $\langle \psi_k | \hat{H} | \psi_{k'} \rangle$ and $\langle \phi_k | \hat{H} | \phi_{k'} \rangle$.

$$\langle \psi_k | \hat{H} | \psi_{k'} \rangle = \frac{1}{N} \sum_{mm'} e^{i(kR_m - k'R_{m'})} \langle \psi_m | \hat{H} | \psi_{m'} \rangle = \frac{\varepsilon_A}{N} \sum_{mm'} e^{i(kR_m - k'R_{m'})} \delta_{mm'}$$
$$\frac{\varepsilon_A}{N} \sum_{m} e^{i(k-k')R_m} = \varepsilon_A \delta_{kk'} \quad (1)$$

and the same procedure for ϕ_k gives

$$\langle \phi_k | \hat{H} | \phi_{k'} \rangle = \varepsilon_B \delta_{kk'} \quad (2)$$

The energy spectrum is independent of k so it is simply the constants ε_A and ε_B:



2.

With $t > 0$ we need to calculate the elements $\langle \psi_k | \hat{H} | \phi_k \rangle$. The lattice constant is $a = 2c$.

$$\langle \psi_k | \hat{H} | \phi_k \rangle = \frac{1}{N} \sum_{mn} e^{-ik(R_m - R_n)} \langle \psi_m | \hat{H} | \phi_n \rangle =$$
$$-\frac{t}{N} \sum_{mn} e^{-ik(R_m - R_n)} (\delta_{R_m, R_{n+a/2}} + \delta_{R_m, R_{n-a/2}}) =$$
$$-\frac{t}{N} \sum_{n} \left(e^{ika/2} + e^{-ika/2} \right) = -2t \cos(kc) \quad (3)$$

so the matrix representation of $\hat{H}$ is block diagonal with each block given by

$$H = \begin{pmatrix} \varepsilon_A & -2t\cos(ka/2) \\ -2t\cos(ka/2) & \varepsilon_B \end{pmatrix}. \quad (4)$$

3.
Diagonalizing eq. 4 gives the two energy bands

$$E_{\pm} = (\varepsilon_A + \varepsilon_B)/2 \pm \left[(\varepsilon_A - \varepsilon_B)^2/4 + 4t^2 \cos^2(ka/2)\right]^{1/2} \tag{5}$$

4.
The energy bands are plotted here for $\varepsilon_A + \varepsilon_B = 0$ and $\varepsilon_A - \varepsilon_B = t = 1\,\mathrm{eV}$: