7 Radiation of relativistic particles

7.1 General properties of relativistic radiation sources

The wave 4-vector is $K^\alpha = (\omega/c, \mathbf{k}) = (\omega/c)(1, \mathbf{n})$. The Lorentz transformation, $K' = \hat{\Lambda} K$, to the frame moving with the velocity v in the x direction is presented by the matrix

$$\hat{\Lambda} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. The reverse matrix describing the transformation from the moving to the lab frame,$K = \hat{\Lambda}^{-1} K'$, is obtained by replacing v by $-v$ in Λ.

Doppler effect. If light is emitted by a moving source, the frequency detected in the lab frame is

$$\nu = \frac{\nu'}{\gamma(1 - \beta\cos\theta)}, \tag{7.2}$$

where θ is the angle between the ray and the direction of motion (in the lab frame). Specifically if the source moves towards the observer, $\nu = \gamma(1+\beta)\nu'$.

Aberration and relativistic beaming. If a ray is emitted at the angle θ' in the source frame, the emission angle in the lab frame is

$$\tan\theta = \frac{k_2}{k_1} = \frac{k_2'}{\gamma(k_1' + \beta k_0')} = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}. \tag{7.3}$$

One sees that the radiation from a highly relativistic source, $\gamma \gg 1$, is beamed in a forward cone of opening angle $\theta \sim 1/\gamma$.

Transit time effect. If the source is moving near the speed of light along a direction which lies close to the line of sight, then the source almost catches up with its own radiation but not quite. Consider an object which moves from a point P_1 to a point P_2 in a time Δt in the observers frame. If the angle between the direction of motion and the line of sight is θ, the time difference between the time of receptions of photons emitted at P_1 and P_2 is

$$\Delta t_{\rm rec} = \Delta t(1 - \beta\cos\theta). \tag{7.4}$$

Specifically for a highly relativistic source moving towards the observer one gets $\Delta t_{\rm rec} = \Delta t/2\gamma^2$. This effect can give the illusion of apparent transverse motion which is greater than the speed of light. The source shifts in the plane of the sky by $L_\perp = v\Delta t \sin\theta$. Therefore the apparent transverse velocity,

$$\beta_{\rm app} = \frac{L_\perp}{c\Delta t_{\rm rec}} = \frac{\beta\sin\theta}{1 - \beta\cos\theta}, \tag{7.5}$$

could be as large as $\gamma\beta$.

Lorenz invariancy of the radiation power. In most cases the emission is symmetric in the sense that the energy emitted in the direction $\mathbf{n}$ is the same as in $-\mathbf{n}$. Then the momentum loss in the comoving frame of the source is zero. Let the source radiate the energy $\Delta E'$ during the time $\Delta t'$ in the comoving frame. The corresponding energy-momentum 4-vector is $(\Delta E', 0)$;

the displacement 4-vector is $(\Delta t, 0)$. Both these 4-vectors are transformed to the observer's frame according to the Lorentz transformation to yield $\Delta E = \gamma \Delta E'$; $\Delta t = \gamma \Delta t'$. Now one can see that the radiation power is a relativistic invariant,

$$\frac{dE}{dt} = \frac{dE'}{dt'}. \tag{7.6}$$

7.2 Synchrotron radiation

A non-relativistic electron rotates in the magnetic field with the Larmor frequency

$$\nu_B = \frac{eB}{2\pi m_e c} = 2.8 \cdot 10^6 B \,\mathrm{Hz} \tag{7.7}$$

where B is in Gauss. Non-relativistic electrons radiate predominantly at the Larmor frequency and a few lower harmonics.

The gyration frequency of a relativistic electron is

$$\tilde{\nu}_B = \frac{\nu_B}{\gamma}. \tag{7.8}$$

Radiation of a highly relativistic electron is beamed to within the angle $\sim 1/\gamma$ therefore only the radiation emitted during the fraction $1/\gamma$ of the gyration period is received by an observer. Because of the transit time effect, the observed duration of the pulse is even γ^2 times smaller so that the observer sees pulses of the duration

$$\Delta t \sim \frac{1}{\gamma^2} \frac{1}{\tilde{\nu}_B \gamma} = \frac{1}{\gamma^2 \nu_B}, \tag{7.9}$$

the interval between the pulses being $1/\tilde{\nu}_B$. The maximum Fourier component of the spectral decomposition of the observed pulse corresponds to the reciprocal of the pulse duration

$$\nu_m = \nu_B \gamma^2. \tag{7.10}$$

The detailed calculation shows that a relativistic electron in the magnetic field radiates the continuum spectrum with the maximum close to $0.5\nu_m \sin\theta$, where θ is the electron pitch angle. At $\nu \ll \nu_m$, the spectrum is proportional to $\nu^{1/3}$. At $\nu \gg \nu_m$, the power decreases exponentially.

Taking into account that the total radiation power is a relativistic invariant, one can write

$$P = \frac{2}{3} \frac{e^2 a'^2}{c^3}, \tag{7.11}$$

where a' is the electron acceleration in the comoving frame. In the lab frame, $a = 2\pi \tilde{\nu}_b c \sin\theta = eB \sin\theta / m_e \gamma$. In order to perform transformation, let us define the 4-vector of acceleration

$$W = \frac{dU}{ds} = \frac{\gamma}{c} \left(\frac{d\gamma}{dt}, \frac{d\gamma \mathbf{v}}{cdt} \right), \tag{7.12}$$

where $U = dx/ds = \gamma(1, \mathbf{v}/c)$ is the velocity 4-vector; $ds = \sqrt{c^2 dt^2 - d\mathbf{x}^2} = cdt/\gamma$ the interval. In our case, $\mathbf{a} \cdot \mathbf{v} = 0$, therefore $d\gamma/dt = 0$; this implies $W = (\gamma/c)^2 (0, \mathbf{a})$. In the comoving frame, $W' = (0, \mathbf{a}'/c^2)$, therefore writing out the relativistic invariant W^2 in both frames yields $a'^2 = \gamma^4 a^2$. Now one finally finds

$$P = \frac{2}{3}\frac{e^4 B^2}{m_e^2 c^3}\gamma^2 \sin^2\theta = 2\sigma_T U_B c\gamma^2 \sin^2\theta, \tag{7.13}$$

where $U_B = B^2/8\pi$ is the magnetic energy density. In a source with the chaotic magnetic field, $\overline{\sin^2\theta} = 2/3$, therefore the average energy loss rate is

$$P == \frac{4}{3}\sigma_T U_B c\gamma^2. \tag{7.14}$$

The relativistic particles typically have a power-law distribution

$$N(\gamma)d\gamma = K\gamma^{-p}d\gamma. \tag{7.15}$$

In most cases, $p = 2 \div 3$. Each electron radiates mostly at $\nu \sim \nu_m$. Since the radiation spectrum is narrower than the breadth of the power-law electron energy spectrum, one can assume that an electron of energy $E = m_e c^2\gamma$ radiates away its energy at $\nu = \nu_m \sin\theta = \gamma^2\nu_B \sin\theta$. Assuming isotropic angular distribution of electrons, one can write the energy radiated in the frequency range ν to $\nu + d\nu$ in the unit solid angle in the form

$$j_\nu d\nu = \frac{P}{4\pi}N(\gamma)d\gamma, \tag{7.16}$$

where

$$\gamma = \left(\frac{\nu}{\nu_B \sin\theta}\right)^{1/2}; \qquad d\gamma = \frac{d\nu}{2(\nu_B\nu\sin\theta)^{1/2}}. \tag{7.17}$$

Now θ is the angle between the magnetic field and the line of sight. Finally one gets

$$j_\nu = \frac{\sigma_T c K U_b \sin\theta}{4\pi\nu_B}\left(\frac{\nu}{\nu_B\sin\theta}\right)^{-\alpha}; \qquad \alpha = \frac{p-1}{2}. \tag{7.18}$$

There is an absorption process corresponding to any emission process. In principle, one can find the synchrotron absorption coefficient from the detailed balance. A simpler way to take into account the synchrotron absorption within the source (often called self-absorption) is to notice that the radiation intensity from any body cannot exceed the blackbody intensity, $B_\nu = 2kT(\nu/c)^2$, whose temperature is determined by the electron energy; in our case $kT \sim \gamma m_e c^2$. Taking into account the relation (7.17) between the electron energy and the emitted frequency, one gets the maximal possible intensity of the synchrotron radiation

$$I_\nu = \left(\frac{8\pi m_e^3 c}{eB}\nu^5\right)^{1/2}. \tag{7.19}$$

Note that this intensity depends only on the magnetic field within the source.

If a source of the size R is transparent, the intensity of the emitted radiation is just $I_\nu \sim j_\nu R$. Typically j_ν grows with the decreasing frequency (see, e.g., eq. (7.18) for the power-law electron spectra). In this case, the source becomes opaque at small frequencies and then the intensity is described by eq. (7.19). The transition frequency, ν_{abs}, is determined by the condition

$$\left(\frac{8\pi m_e^3 c}{eB}\nu_{abs}^5\right)^{1/2} = j_{\nu_{abs}} R. \tag{7.20}$$

7.3 Compton scattering

Kinematics of the Compton scattering is described by the conservation of 4-momentum:

$$P + \hbar K = P_1 + \hbar K_1. \tag{7.21}$$

Here $P = m_e c U$, $P^2 = m_e^2 c^2$; $K^2 = 0$. In order to find the final state of the photon, one has to move the term with K_1 to the lhs and square the obtained equation:

$$(P + \hbar K - \hbar K')^2 = m_e^2 c^2. \tag{7.22}$$

If the electron was initially at rest, $P = (m_e c, 0)$, one gets

$$\nu_1 = \frac{\nu}{1 + (h\nu/m_e c^2)(1 - \cos\theta)}, \tag{7.23}$$

where θ is the scattering angle. At $h\nu \ll m_e c^2$, the photon frequency does not change; this is the Thomson scattering.

The cross-section is the Klein-Nishina formula. At $h\nu \ll m_e c^2$, it is reduced to the Thomson cross-section (3.27); in the opposite limit,

$$\sigma = \frac{3}{8}\sigma_T \frac{m_e c^2}{h\nu}\left(\ln\frac{2h\nu}{m_e c^2} + \frac{1}{2}\right). \tag{7.24}$$

Let us now consider the scattering on a moving electron. The electron sees the photon frequency

$$\nu' = \nu\gamma(1 - \beta\cos\theta), \tag{7.25}$$

where θ is the angle between the velocity of the electron and the photon. At $\gamma \gg 1$, one gets typically $\nu' \sim \gamma\nu$. If

$$\gamma h\nu \ll m_e c^2, \tag{7.26}$$

the photon frequency does not change in the scattering process, $\nu_1' = \nu'$. Due to the relativistic beaming, the scattered photon is directed in the lab frame to the direction close to the electron velocity, $\theta_1 \sim \gamma^{-1}$; then eq. (7.2) yields the frequency of the scattered photon in the lab frame:

$$\nu_1 = \frac{\nu_1'}{\gamma(1 - \beta\cos\theta_1)} = \frac{2\nu_1'}{\gamma(\theta_1^2 + 1/\gamma^2)} \sim \gamma\nu_1'. \tag{7.27}$$

Now one sees that the photon frequency increases significantly as a result of scattering

$$\nu_1 \sim \gamma^2\nu. \tag{7.28}$$

This process is called the inverse Compton scattering.

Now let us consider a highly relativistic electron moving through the isotropic radiation field with the energy density $U_{\rm rad}$. Let the average photon energy be ε_0; then the photon number density is $N = U_{\rm rad}/\varepsilon_0$. Due to relativistic beaming, the electron sees the radiation beam directed towards the velocity. The energy flux in the beam is $F' = \varepsilon' N' c$. The photon number density is the zeroth component of the 4-vector $(N, \mathbf{j})$, where $\mathbf{j}$ is the photon flux. In the lab frame, $\mathbf{j} = 0$; therefore the Lorentz transformation yields $N' = \gamma N$. Taking into account that $\varepsilon' \sim \gamma\varepsilon$, one finds $F' \sim \gamma^2 U_{\rm rad} c$. Let us now assume that the typical radiation frequency satisfies the Thomson scattering condition (7.26). Then the scattered photons have the characteristic energy $\varepsilon \sim \gamma^2 \varepsilon_0$. The power of the scattered radiation in the electron frame is $P' = \sigma_T F'$. The Thomson scattering satisfies the forward-backward symmetry condition, which yields the Lorentz invariancy of the radiated power. Therefore one finally finds the power of the scattered radiation as

$$P = \frac{4}{3}\sigma_T c U_{\rm rad}\gamma^2, \tag{7.29}$$

where the factor 4/3 is obtained in rigorous calculation. Note the similarity between the Compton power and the synchrotron power (7.14).

7.4 Effect of energy losses on the particle and the emission spectra

The total synchrotron and Compton energy loss rate is presented as

$$P = \frac{4}{3}\sigma_T c U \gamma^2, \tag{7.30}$$

where $U = U_B + U_{\rm rad}$ is the energy density of the background magnetic field and radiation. A characteristic time for an electron with the Lorentz factor γ to loose its energy is estimated as

$$t_{\rm loss} = \frac{m_e c^2 \gamma}{P} = \frac{3 m_e c}{4\sigma_T U \gamma}. \tag{7.31}$$

If relativistic electrons were instantaneously injected into the source, after the time t, only electrons with the Lorentz factors less than

$$\gamma_0 = \frac{3 m_e c}{4\sigma_T U t} \tag{7.32}$$

survive so that an exponential turnover should be observed in the synchrotron spectrum at the frequency

$$\nu_{\rm br} = \nu_B \gamma_0^2, \tag{7.33}$$

Let now the relativistic electrons be continuously injected into the source with the power-law spectrum

$$Q = \frac{dN}{d\gamma dt} = q\gamma^{-p}. \tag{7.34}$$

If the life time of the source is t, all the electrons injected with the energy less than γ_0 are accumulated in the system whereas at larger energy, only the electrons injected during the time $t_{\rm loss}$ remain so that the total electron spectrum has a break at the energy γ_0:

$$\frac{dN}{dE} = \begin{cases} tQ = tq\gamma^{-p}; & \gamma < \gamma_0; \\ t_{\rm loss}Q = \frac{3m_e c}{4\sigma_T U} q\gamma^{-(p+1)}; & \gamma > \gamma_0. \end{cases} \tag{7.35}$$

Correspondingly, the observed synchrotron spectrum ha a break at the frequency $\nu_{\rm br}$, at which the spectral slope increases by 0.5, from $\alpha = (p-1)/2$ to $\alpha = (p+1)/2$.