QFT take home exam. Submission date: 14.02.2011

1. Consider an $O(N)$-symmetric $(\phi^2)^2$ theory with euclidean action in $d = 4 - \epsilon$ dimensions. Here ϕ is an N-component scalar field.

A). Define the Feynman rules. Each Feynman diagram is proportional to N to some power. Order the diagrams according to their large N-counting.
B). Compute the β function of the coupling. Find the Wilson-Fisher fixed point.
C). Use the RG flow of the mass operator in order to find the critical exponent ν for the correlation length near the critical point.

2. In $d = 4-\epsilon$ dimensions consider an $O(N)$-symmetric theory with euclidean action for N component scalar field ϕ:

$$S[\phi] = \int d^d x [\frac{1}{2}(\partial_\mu \phi) + N\, U(\phi^2/N)]$$

where U is a general polynomial potential. An implicit cutoff Λ is assumed. Large N clustering property suggests to take $\rho = \phi^2/N$ as a new dynamical variable. To this goal we use the Lagrange multiplier technique based on the identity:

$$1 = N \int d\rho \delta(\phi^2 - N\rho) = \frac{N}{4\pi i} \int d\rho d\lambda e^{\lambda(\phi^2 - N\rho)/2}$$

where the λ-integration contour runs parallel to the imaginary axis.

A). Write the action for ϕ, ρ, λ fields.
B). Split the field ϕ into one component field σ and $N-1$ fields π. Derive an effective action for the fields σ, ρ, λ by integrating out the fields π.
C). Consider the large N limit. Assume $\sigma = O(N^{1/2})$, $\rho = O(1)$, $\lambda = O(1)$. Derive saddle point equations for the fields. The ground state solution can be considered as space independent $\sigma(x) = \sigma$; $\rho(x) = \rho$; $\lambda(x) = m^2$ (note that λ provides mass to the ϕ field).
D). Find two solutions of the above equations, corresponding to broken and symmetric phases. These are low and high temperature phases. Define conditions for the critical point. Identify the order parameter and the Goldstone bosons.
E). For the potential

$$U(\rho) = \frac{1}{2} r\rho + \frac{u}{4!}\rho^2$$

find the "critical temperature" r_c.
F). Consider deviation from the critical point $\tau = r - r_c$. In the low temperature phase

$$\sigma \sim (-\tau)^{2\beta}$$

Find the critical exponent β.
In the high temperature phase, the correlation length

$$\xi \sim m^{-1} \sim \tau^{-\nu}.$$

Find the critical exponent ν. Compare with the result of the problem 1. Explain.