Physics 3 for physics - Quiz 2006/7 - solution

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Question 1:
The mass movement along the y axis produces a wave in the string propagating in the $+x$ direction. The force applied by the string to the mass is

$$F = T\frac{\partial y}{\partial x}$$

For a wave propagating in the $+x$ direction $y(x,t) = A\mathrm{e}^{\imath(\omega t - kx)}$ the force can be written as

$$F = -Z\frac{\partial y}{\partial t}$$

where y is the transversal displacement of the string, and $Z = \sqrt{\rho T} = 5$ kg/sec is the impedance of the string. It can be seen that this force acts on the mass as a friction force which is proportional to the velocity of the mass.

The solution of

$$\frac{\partial^2 y}{\partial t^2} + \Gamma\frac{\partial y}{\partial t} + \omega_0^2 y = 0$$

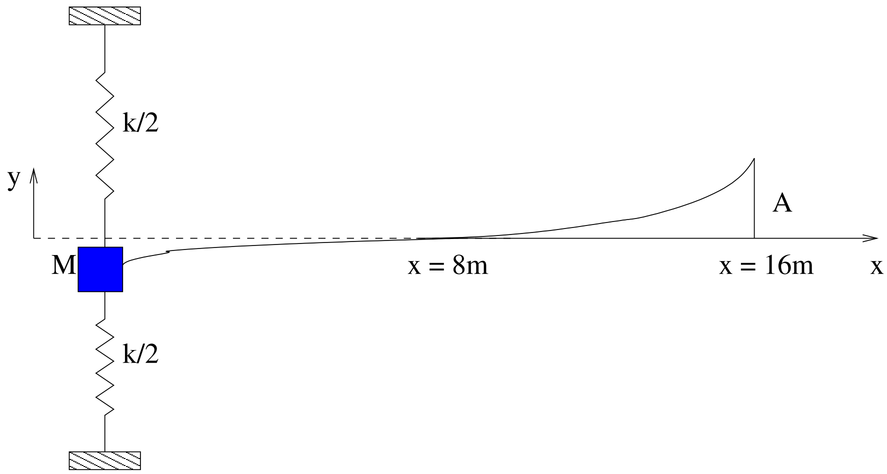
is

$$y(t) = A\mathrm{e}^{-\frac{\Gamma}{2}t}\cos(\omega t + \phi)$$

where $\omega^2 = \omega_0^2 - \left(\frac{\Gamma}{2}\right)^2$.
In our case $\Gamma = \frac{Z}{M} = 2.5\ \mathrm{sec}^{-1}$ and $\omega = \sqrt{\frac{k}{M} - \left(\frac{\Gamma}{2}\right)^2} = \sqrt{\frac{32}{2} - \left(\frac{2.5}{2}\right)^2} \simeq 3.8\ \mathrm{sec}^{-1}$. So that the period is: $T = \frac{2\pi}{\omega} \simeq 1.65$sec.
The amplitude lowers to the $1/e$ of its initial value for $\mathrm{e}^{-\frac{\Gamma}{2}t} = \mathrm{e}^{-1}$. In our case it happens at $t = \frac{2}{\Gamma} = \frac{2M}{Z} = 0.8$ sec.
The phase velocity in the string equals to the group velocity.

$$v = \sqrt{\frac{T}{\rho}} = 20 \text{ m/s}$$

So the pulse travels $z = vt = 16$ m
Since the force $F = -Z\frac{\partial y}{\partial t}$ decays with time, the amplitude of the produced wave at $z = 0$ decays, so that at larger z the amplitude is larger (since it was generated at earlier times).
We are interested at the times $t \lesssim T/2$. In the first quarter of the period the string is above the x axis and and it starts moving from A and in the second quarter the string is below the x axis and the largest amplitude is A/e.



In order to find the the propagating wave $y(x,t)$ in the string, first of all we write the boundary conditions

$$y(x=0,t) = Be^{-\frac{\Gamma}{2}t}\cos(\omega t + \phi)$$

At $t = 0$ the mass is at $A = 0.05$:

$$y(x=0,t=0) = Be^{-\frac{\Gamma}{2}t}\cos(\omega t + \phi)|_{t=0} = B\cos\phi = A$$

and its velocity is zero:

$$\dot{y}(x=0,t=0) = Be^{-\frac{\Gamma}{2}t}\left(-\frac{\Gamma}{2}\cos(\omega t + \phi) - \omega\sin(\omega t + \phi)\right)|_{t=0} = B\left(-\frac{\Gamma}{2}\cos\phi - \omega\sin\phi\right) = 0$$

So that

$$\tan\phi = -\frac{\Gamma}{2\omega}$$
$$\cos^2\phi = \frac{1}{\tan^2\phi + 1}$$

In non-dispersive medium the wave should satisfy the d'Alembert solution of the wave equation $y(x,t) = y(x - vt)$ for the wave propagating in the positive direction. Or, in other words, if $t' = t - \frac{x}{v}$ then $y(x,t) = y(0,t')$. Therefore,

$$y(x,t) = A\sqrt{1 + \left(\frac{\Gamma}{2\omega}\right)^2}\ e^{-\frac{\Gamma}{2}\left(t - \frac{x}{v}\right)}\cos\left(\omega t - kx - \arctan\left(\frac{\Gamma}{2\omega}\right)\right)$$

where $\omega = vk$. Substituting the numerical values into the expression we get

$$y(x,t) \simeq 0.053\ e^{-1.25\left(t - \frac{x}{20}\right)}\cos(3.8t - 0.21x - 0.31)$$

Question 2:

The unstretched springs have a length L.
After stretching the springs the tension in the spring **A** is

$$T_A = k_A(2L - L) = k_A L$$

and the tension in the spring **B** is

$$T_B = k_B(3L - L) = 2k_B L$$

Since the tensions should be equal at the attachment point $k_A = 2k_B$.

Transversal vibrations
After stretching the springs the mass densities are:

$$\rho_A = \frac{M}{2L}, \quad \rho_B = \frac{M}{3L}$$

So that $\rho_A = \frac{3}{2}\rho_B$. The impedance for transversal vibrations is $Z = \sqrt{\rho T}$. Then

$$Z_A = \sqrt{\rho_A T} = \sqrt{\frac{3}{2}\rho_B T} = \sqrt{\frac{3}{2}} Z_B$$

The reflection coefficient is:

$$R = \frac{Z_A - Z_B}{Z_A + Z_B} = \frac{\sqrt{\frac{3}{2}} - 1}{\sqrt{\frac{3}{2}} + 1} \simeq 0.101$$

The trasmission coefficient is:

$$T = 1 + R \simeq 1.101$$

Longitudinal vibrations
For longitudinal vibrations the impedance is $Z = \sqrt{KM}$.

$$Z = \sqrt{KM} = \sqrt{(Ka) \times (M/a)} = \sqrt{\rho T}$$

where a is the length of a spring of mass M and the constant K. For the two springs T is equal, but ρ is different (like in the previous case).
Therefore, the reflection and transmission coefficients are the same for longitudinal and transversal oscillations.

$$R = \frac{Z_A - Z_B}{Z_A + Z_B} = \simeq 0.101, \quad T = 1 + R \simeq 1.101$$

Comment:
The reflection coefficient (and the impedance) can be derived in the following way:
Write the dispersion relation for a chain of masses M connected by springs K ($\omega = 2\sqrt{\frac{K}{M}} \sin\left(\frac{1}{2}ka\right)$).
Now suppose there two such chains connected at $x = 0$: one chain with the coefficient K_1 and the second one with K_2. Assume a harmonic wave coming from the right, write the equation of motion for the mass connecting the chains and solve it in the continuous limit $a \to 0$. This leads to the formula for the reflection coefficient R.