Physics3 exam solutions 26.1.11

1. Two masses under a force

   (a) The equations of motion The spring constant is the y component of tension $= T\frac{\text{change in } y}{L}$, hence

$$\begin{aligned} M\ddot{y}_1 &= -\frac{T}{L}y_1 + \frac{T}{L}(y_2 - y_1) \\ M\ddot{y}_2 &= -\frac{T}{L}(y_2 - y_1) + \frac{T}{L}(h_0\cos(\omega t) - y_2) \end{aligned} \tag{1}$$

   Choose $y_1(t) = y_1(\omega)\cos(\omega t)$, $y_2(t) = y_2(\omega)\cos(\omega t)$

$$\begin{aligned} (-M\omega^2 + 2\frac{T}{L})y_1(\omega) - \frac{T}{L}y_2(\omega) &= 0 \\ -\frac{T}{L}y_1(\omega) + (-M\omega^3 + 2\frac{T}{L})y_2(\omega) &= \frac{T}{L}h_0 \end{aligned} \tag{2}$$

   (b) The normal modes for a fixed end, i.e. $h_0 = 0$,

$$\begin{aligned} &\begin{vmatrix} -M\omega^2 + 2\frac{T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & -M\omega^2 + \frac{T}{L} \end{vmatrix} = (-M\omega^2 + 2\frac{T}{L})^2 - (\frac{T}{L})^2 = 0 \\ &\Rightarrow \omega_1 = \frac{1}{M}\sqrt{\frac{T}{L}}, \qquad \omega_1 = \frac{1}{M}\sqrt{\frac{3T}{L}} \end{aligned} \tag{3}$$

   (c) From (a)

$$\begin{aligned} y_2(\omega) &= y_1(\omega)\frac{-M\omega^2 + 2T/L}{T/L} \\ y_1(\omega) &= \frac{(\frac{T}{L})^2 h_0}{(-M\omega^2 + 2\frac{T}{L})^2 - (\frac{T}{L})^2} = \frac{\frac{T}{L}h_0}{M^2(\omega - \omega_1)(\omega - \omega_2)} \end{aligned} \tag{4}$$

   $y_1(\omega)$ has $\pm\infty$ at the normal mode frequencies ω_1 , ω_2. This is a resonance phenomena. $y_2(\omega)$ has the same poles and also changes sign at $\omega = \frac{1}{M}\sqrt{\frac{2T}{L}}$.

   The power supplied by the force, say on y_1, is $\sim \cos(\omega t)\dot{y}_1 = -\frac{1}{2}y_1(\omega)\omega\sin(2\omega t)$ so the sign of $y_1(\omega)$ signifies in which cycle energy is absorbed or released from particle 1. In particular if y_1, y_2 have the same sign they both absorb at the same times, while if they have opposite signs when one absorbs the other releases, and vice versa. The time average of the power is zero at all cases, since there is no friction.

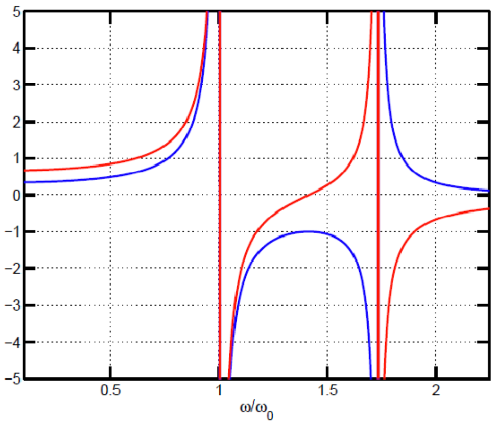


FIG. 1: $y_1(\omega)$ are the lines without a zero (blue thin lines) while $y_2(\omega)$ are the lines with one zero in the middle (red thick lines). Here $\omega_0 = \frac{1}{M}\sqrt{\frac{T}{L}}$.

2. (a) the reflection is $R = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1-1.5}{1+1.5} = -0.2$ hence the intensity reflected is $0.04 I_0$, intensity transmitted is $0.96 I_0$.

The incoming wave is $A[\cos(\omega t), sin(\omega t)]$. The reflected wave has the same R for both components, hence it is $-0.04A[\cos(\omega t), sin(\omega t)]$ which rotates in the same direction hence it is left polarized for the outgoing wave.

(b) The critical angle is $\sin\theta_c = \frac{n_2}{n_1} = \frac{1}{1.5}$ hence $\theta_c \approx 41$ degrees. Therefore the incident radiation on AC is totally reflected.

(c) Reflection is now $R = \frac{1.5-1}{1.5+1} = +0.2$. Hence transmitted intensity is $0.96 I_0 \times 0.96 \approx 0.92 I_0$.

(d) Neglect first interference and use intensity conservation: Since $0.04 \times 0.96 I_0$ is reflected at BC it adds to the transmission at AB

$(0.04 + 0.04 \times 0.96^2) I_0 \approx 0.07 I_0$. The total of reflected and transmitted intensity is $0.99 I_0$ so there is still 1% missing from the next orders in reflections.

With interference: Since the trajectories of all reflected waves are all equal to L, i.e. the length of AB or BC, they can yield a finite interference. If A_0 is the incoming amplitude, k_0 is the wavevector in vacuum and k_1 in glass, then the 1st reflected wave is $-0.2A_0 \cos(\omega t + k_0 x)$. The 2nd reflected wave travels an additional length $2L$ and also undergoes a phase shift ϕ_0 at the reflection from AC (unspecified). This wave is transmitted at AB (factor 1.2), reflected at BC (factor +0.2) and transmitted out of AB (factor 0.8), hence the total reflected wave is

$$\Psi_r = A_0[-0.2\cos(\omega t + k_0 x) + 0.2 \cdot 0.96 \cos(\omega t + k_0 x - 2k_1 L - 2\phi_0)] \qquad (5)$$

Hence an additional interference, depending on parameters k_0, L, ϕ_0 which are unspecified.

3. (a) Consider the scattering between depth h and depth $h - dh$, at a surface in the x direction. Matching of the wave along the surface requires that $k_x = \frac{\omega}{v_\phi} \sin\theta(h)$ is the same on both sides, this leads to Snell's law for the change in propagation direction. Doing this matching successively for all h we have that k_x is conserved, i.e. it is equal at h_1 and at any other h,

$$k_x = \frac{\omega}{\sqrt{\mu}}\sqrt{\alpha h + \rho_0}\sin\theta(h) = \frac{\omega}{\sqrt{\mu}}\sqrt{\alpha h_1 + \rho_0}\sin\theta_0$$

$$\Rightarrow \qquad \sin\theta(h) = \sin\theta_0\sqrt{\frac{\alpha h_1 + \rho_0}{\alpha h + \rho_0}} \tag{6}$$

One can also integrate a differential form for $\sin\theta$, a somewhat longer calculation. When h decreases towards the surface θ is increasing, i.e. the pulse direction approaches the horizontal. If $\sin\theta = 1$ happens at $h > 0$ the pulse cannot reach the surface (it actually starts bending down). The condition for reaching the surface is then

$$\sin\theta(h = 0) < 1 \Rightarrow \qquad \sin\theta_0\sqrt{1 + \frac{\alpha h_1}{\rho_0}} < 1 \tag{7}$$

(b) The wave has at depth h wavevectors $[k_x, k_y] = \frac{\omega}{v_\phi}[sin\theta(h), \cos\theta(h)]$ (actually h and y are the same, $k_y = k_h$). The solution for the elastic distortion $u(x, h)$ (could be either transverse or longitudinal) is equal on both sides of a surface in the x direction, hence $\frac{\partial u}{\partial x}$ is also equal on both sides so that k_x is constant, h independent. The possible discontinuity in $\frac{\partial u}{\partial h}$ reflects the balance of forces $\mu(h)\frac{\partial u}{\partial y}$ on the surface, assuming in general $\mu(h)$. Hence $\mu(h)k_y(h)$ is $\sim$force, i.e. the analog of Tk in the elastic string problem. Hence $Z = \frac{Tk}{\omega}$ corresponds to the ratio of force/(physical velocity), i.e.

$$Z(h) = \frac{\mu}{\omega}k_y(h) \tag{8}$$

(c) First rewrite $Z(h)$ explicitly as function of h,

$$Z(h) = \frac{\mu}{\omega}k_y(h) = \frac{\mu}{\omega}\sqrt{\frac{\omega^2 - v_\phi^2 k_x^2}{v_\phi^2}} = \sqrt{\mu(\alpha h + \rho_0) - (\frac{\mu k_x}{\omega})^2} \tag{9}$$

$$Z(h - dh) = Z(h) - \mu\alpha\frac{dh}{2\sqrt{\mu(\alpha h + \rho_0) - (\frac{\mu k_x}{\omega})^2}} \tag{10}$$

The transmission from h to $h - dh$ is then

$$T = 1 + \frac{Z(h) - Z(h - dh)}{Z(h) + Z(h - dh)} = 1 + \frac{\mu\alpha dh}{4(\mu(\alpha h + \rho_0) - (\frac{\mu k_x}{\omega})^2)} \tag{11}$$

The change in the amplitude $A(h)$ is

$$dA = TA - A = \frac{\mu\alpha dh}{4(\mu(\alpha h + \rho_0) - (\frac{\mu k_x}{\omega})^2)} A \tag{12}$$

Integration yields

$$\ln \frac{A(h)}{A_1} = \frac{1}{4} \ln \frac{\mu(\alpha h + \rho_0) - (\frac{\mu k_x}{\omega})^2}{\mu(\alpha h_1 + \rho_0) - (\frac{\mu k_x}{\omega})^2}$$

$$\Rightarrow A(h) = A_1 \left[\frac{\alpha h + \rho_0 - (\alpha h_1 + \rho_0)\sin^2\theta_0}{(\alpha h_1 + \rho_0)(1 - \sin^2\theta_0)} \right]^{1/4} \tag{13}$$

For small $\rho_0\,, \sin\theta_0$ we have that $A(h)$ decrease as the pulse propagates upwards towards the surface as $A(h) = A_1(h/h_1)^{1/4}$.

4. (a) The scattering function is , with $k_x = k\sin\theta = \frac{2\pi}{\lambda}\sin\theta$.

$$F(k_x) = \sum_{n=0}^{N-1} \mathrm{e}^{ik_x nd} = \frac{1-\mathrm{e}^{ik_x Nd}}{1-\mathrm{e}^{ik_x d}} \tag{14}$$

Hence the scattering intensity is

$$I(\theta) = I(0)\frac{\sin^2(\frac{1}{2}k_x Nd)}{N^2\sin^2(\frac{1}{2}k_x d)} \tag{15}$$

The normalization N^2 is obtained by taking the limit $\frac{1}{2}k_x d \to 0$.

(b) The main peaks correspond to the zeroes of the denominator at $\frac{1}{2}k_x d = \frac{\pi}{\lambda}d\sin\theta = n\pi$ with n integer. For small θ (where Huygens principle is valid) these are $\theta_n = \frac{\lambda}{d}n$. The intensities are found as a limit,

$$I(\theta_n) = \frac{I(0)}{N^2}\left[\frac{\frac{\partial}{\partial k_x}\sin\frac{1}{2}k_x Nd}{\frac{\partial}{\partial k_x}\sin\frac{1}{2}k_x d}\right]^2 = I(0) \qquad \theta = \theta_n \tag{16}$$

The width $\Delta\theta$ is obtained by identifying the next zero of the nominator where the denominator does not vanish,

$$\frac{1}{2}k_x Nd \approx \frac{\pi d}{\lambda}(\theta_n \pm \tfrac{1}{2}\Delta\theta) = \pi n \pm \pi$$
$$\Rightarrow \Delta\theta = \frac{2\lambda}{Nd} \tag{17}$$

As $N \to \infty$ the width vanishes and $I(\theta)$ vanishes except at the points $k_x = \frac{2\pi}{d}n$ where $I(\theta_n) = I(0)$. (Bragg scattering).

(c) The scattering function is now, by changing variable $x' = x - nd$,

$$F(\theta) = \sum_{n=0}^{N-1}\int_{nd-\frac{1}{2}D}^{nd+\frac{1}{2}D} \mathrm{e}^{ik_x x}dx = \sum_{n=0}^{N-1}\mathrm{e}^{ik_x nd}\int_{-\frac{1}{2}D}^{\frac{1}{2}D}\mathrm{e}^{ik'_x x}dx' = \frac{1-\mathrm{e}^{ik_x Nd}}{1-\mathrm{e}^{ik_x d}}\cdot\frac{\sin(\frac{1}{2}k_x D)}{\frac{1}{2}k_x} \tag{18}$$

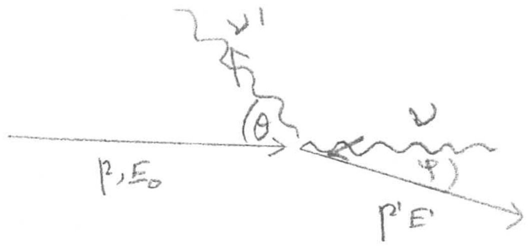
The intensity is then

$$I(\theta) = I(0)\frac{\sin^2(\frac{1}{2}k_x Nd)}{N^2\sin^2(\frac{1}{2}k_x d)}\cdot\frac{\sin^2(\frac{1}{2}k_x D)}{(\frac{1}{2}k_x D)^2} \tag{19}$$

and normalization is again checked by the limit $k_x \to 0$.

The width due to the 2nd factor is $\sim \lambda/D$ which is much larger then that of (b), since $D \ll Nd$. Hence there is no effect of D on the width of scattering. Even the intensities are not affected for the first peaks, i.e. when $\frac{\pi D}{\lambda}\theta_n = \frac{\pi D n}{d} \ll \pi$, i.e. for $n \ll \frac{d}{D}$.

פתרון לשאלה 5



באופן הכללי כותבים:

יש שלוש עבור 3
המשוואות האלה,
קל לפתור
גם נעלמים

$$\overbrace{\sqrt{m^2c^4+p^2c^2}}^{E_0}+h\nu=\sqrt{m^2c^4+p'^2c^2}+h\nu' \quad (1)$$ שימור אנרגיה

$$\frac{h\nu'}{c}\sin\theta=p'\sin\varphi \quad (2)$$ שימור תנע, ציר y

$$p-\frac{h\nu}{c}=p'\cos\varphi-\frac{h\nu'}{c}\cos\theta \quad (3)$$ שימור תנע, ציר x

$$(3)\rightarrow \quad \left(p-\frac{h\nu}{c}+\frac{h\nu'}{c}\cos\theta\right)^2=p'^2\cos^2\varphi$$

$$(2) \quad \left(\frac{h\nu'}{c}\sin\theta\right)^2=p'^2\sin^2\varphi \qquad +$$

$$\Rightarrow (4) \quad p^2+\left(\frac{h\nu}{c}\right)^2+\left(\frac{h\nu'}{c}\right)^2-\frac{2h\nu}{c}p-\frac{2h\nu h\nu'}{c^2}\cos\theta+\frac{2h\nu'}{c}p\cos\theta=p'^2$$

$$(1)\left(\sqrt{m^2c^4+p^2c^2}+h\nu-h\nu'\right)^2=m^2c^4+p'^2c^2$$

$$\Rightarrow (5) \quad m^2c^4+p^2c^2+(h\nu)^2+(h\nu')^2+2E_0h\nu-2E_0h\nu'-2h\nu h\nu'=m^2c^4+p'^2c^2$$

$$\frac{(5)}{c^2}-(4)\Rightarrow \quad \frac{2E_0h\nu}{c^2}-\frac{2E_0h\nu'}{c^2}-\frac{2h\nu h\nu'}{c^2}+\frac{2h\nu}{c}p+\left(\frac{2h\nu}{c}-2p\right)\frac{h\nu'}{c}\cos\theta=0$$

$$\Rightarrow h\nu'=\frac{h\nu(E_0+pc)}{E_0+h\nu+(pc-h\nu)\cos\theta}$$

כיוון שלפי תנאי השאלה $pc-h\nu>0$ המקסימום של אנרגיית הפוטון אחרי ההתנגשות מושג כאשר $\cos\theta=-1$ או $\theta=2\pi$ (ההחזרה מושלמת). במקרה כזה

$$(h\nu')_{max}=\frac{h\nu(E_0+pc)}{E_0-pc+2h\nu}$$