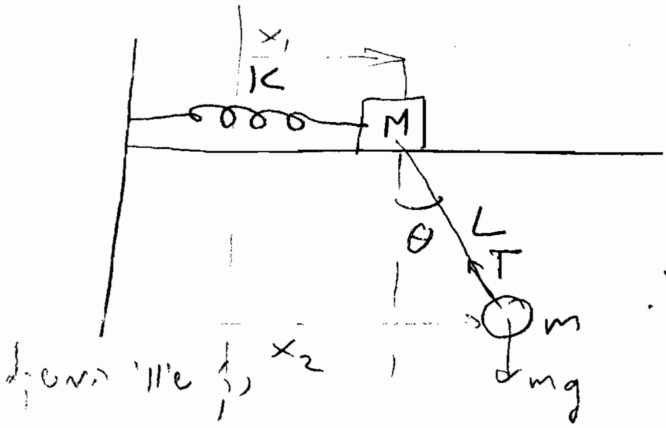


1. בהנחה של $\theta \ll 1$

$T = mg$

ותנועת המסה m היא על ציר האופקי.

$r_{1,2}$ - מיקום המסות על ציר ה-x.

(א)

$$\theta = \frac{x_2 - x_1}{L}$$

$$\vec{r}_1 = x_1 \hat{x}$$

$$\vec{r}_2 = (x_1 + L\sin\theta)\hat{x} \simeq (x_1 + L\theta)\hat{x}$$

$$\ddot{r}_1 = \ddot{x}_1$$

$$\ddot{r}_2 = \ddot{x}_1 + L\ddot{\theta} = \ddot{x}_1 + \ddot{x}_2 - \ddot{x}_1 = \ddot{x}_2$$

$$\Rightarrow \begin{cases} M\ddot{x}_1 = -kx_1 + mg\theta = -kx_1 + mg\dfrac{x_2 - x_1}{L} \\ m\ddot{x}_2 = -mg\theta = -mg\dfrac{x_2 - x_1}{L} \end{cases}$$

(ב) $M = m$

$$\begin{cases} m\ddot{x}_1 = -kx_1 + m\dfrac{g}{L}(x_2 - x_1) \\ m\ddot{x}_2 = -mg\dfrac{1}{L}(x_2 - x_1) \end{cases}$$

$$\begin{cases} \ddot{x}_1 = -\omega_s^2 x_1 + \omega_p^2 x_2 - \omega_p^2 x_1 \\ \ddot{x}_2 = -\omega_p^2 x_2 + \omega_p^2 x_1 \end{cases}$$

$$\omega_p^2 \equiv g/L$$
$$\omega_s^2 = k/m$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} \omega_s^2 + \omega_p^2 & -\omega_p^2 \\ -\omega_p^2 & \omega_p^2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{vmatrix} \omega_s^2 + \omega_p^2 - \omega^2 & -\omega_p^2 \\ -\omega_p^2 & \omega_p^2 - \omega^2 \end{vmatrix} = 0 = (\omega_s^2 + \omega_p^2 - \omega^2)(\omega_p^2 - \omega^2) - \omega_p^4 =$$

$$= \omega_s^2\omega_p^2 - \omega_s^2\omega^2 + \cancel{\omega_p^4} - \omega_p^2\omega^2 - \omega_p^2\omega^2 + \omega^4 - \cancel{\omega_p^4} = 0$$

$$\omega^4 - \omega^2(2\omega_p^2 + \omega_s^2) + \omega_s^2\omega_p^2 = 0$$

$$\omega_{1,2}^2 = \frac{2\omega_p^2 + \omega_s^2 \pm \sqrt{4\omega_p^4 + \cancel{4\omega_p^2\omega_s^2} + \omega_s^4 - \cancel{4\omega_s^2\omega_p^2}}}{2} =$$

$$= \frac{1}{2}\left(2\omega_p^2 + \omega_s^2 \pm \sqrt{4\omega_p^4 + \omega_s^4}\right)$$

$$\omega_1^2 = \omega_p^2 + \frac{1}{2}\omega_s^2 + \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4}$$

$$\begin{pmatrix} \frac{1}{2}\omega_s^2 - \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4} & -\omega_p^2 \\ -\omega_p^2 & -\frac{1}{2}\omega_s^2 - \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \dfrac{-\frac{1}{2}\omega_s^2 - \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4}}{\omega_p^2} \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \omega_p^2 + \frac{1}{2}\omega_s^2 - \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \dfrac{-\frac{1}{2}\omega_s^2 + \frac{1}{2}\sqrt{4\omega_p^4 + \omega_s^4}}{\omega_p^2} \\ 1 \end{pmatrix}$$

$\omega_p \gg \omega_s$ d

$$\omega_1^2 \simeq 2\omega_p^2$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = 0$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{E}_i = A e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

$$I_0 = \langle \vec{E}\cdot\vec{E}^* \rangle = A^2$$

$$k = \frac{2\pi}{\lambda}$$

$$E_R = A e^{i\omega t}\left(2e^{-ik\sqrt{R^2+b^2}} + e^{-ik(R+b)}\right) =$$

$$= A e^{i\omega t}\left(2e^{-ikR\sqrt{1+\left(\frac{b}{R}\right)^2}} + e^{-ik(R+b)}\right) =$$

$$\simeq A e^{i\omega t}\left(2e^{-ikR} e^{-ik\frac{1}{2}\frac{b^2}{R}} + e^{-ikR}e^{-ikb}\right) =$$

$$= A e^{i(\omega t - kR)}\left(2e^{-\frac{1}{2}ik\frac{b^2}{R}} + e^{-ikb}\right)$$

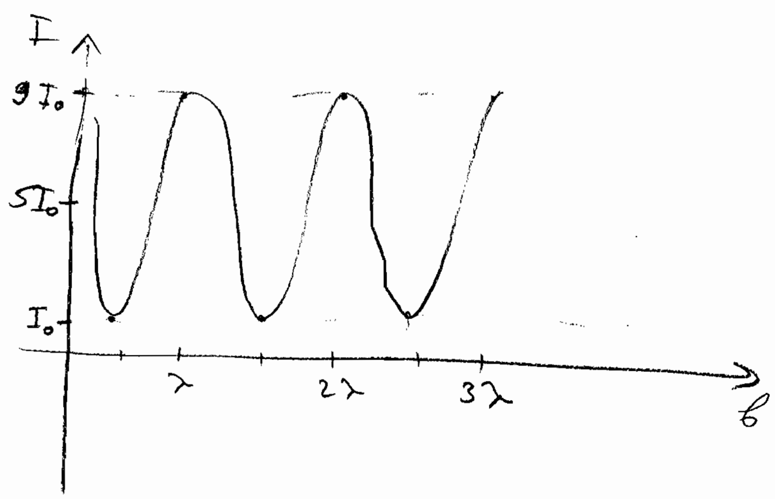
$$\vec{E}_R\cdot\vec{E}_R^* = A^2\left(2e^{-\frac{1}{2}ik\frac{b^2}{R}} + e^{-ikb}\right)\left(2e^{\frac{1}{2}ik\frac{b^2}{R}} + e^{ikb}\right) =$$

$$= A^2\left(4+1+2e^{ikb\left(1-\frac{1}{2}\frac{b}{R}\right)} + 2e^{-ikb\left(1-\frac{1}{2}\frac{b}{R}\right)}\right) =$$

$$= A^2\left(5+4\cos\left[kb\left(1-\frac{1}{2}\frac{b}{R}\right)\right]\right)$$

$$I = I_0\left(5+4\cos\left[kb\left(1-\frac{1}{2}\frac{b}{R}\right)\right]\right) \simeq \qquad b \ll R$$

$$\simeq I_0\left(5+4\cos kb\right) = I_0\left(5+4\cos\frac{2\pi}{\lambda}b\right)$$



max: $b = n\lambda$, $n = 1, 2, \ldots$

min: $b = \frac{1}{2}n\lambda$, $n = 1, 2, \ldots$

ב. בתוך פיסת החומר הדיאלקטרי מהירות האור c/n ולכן

הפרש הפאזה הוא

$$k'b = \frac{\omega}{v} b = \frac{\omega}{c} n b = k n b$$

$$E_R = A e^{i\omega t}\left(2e^{-ik\sqrt{R^2+b^2}} + e^{-ikR - ik'b}\right) \simeq$$

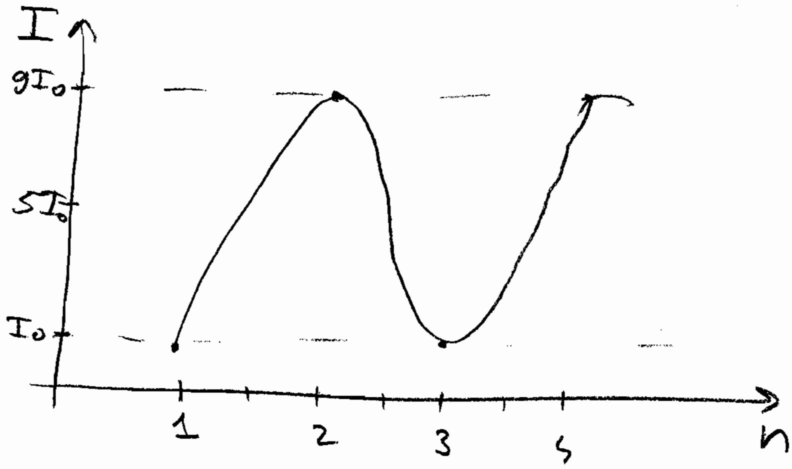
$$\simeq A e^{i(\omega t - kR)}\left(2 + e^{-iknb}\right)$$

$$\vec{E}_R \cdot \vec{E}_R^{\,*} = A^2\left(4 + 1 + 4\cos knb\right)$$

$$I = I_0\left(5 + 4\cos knb\right) = \quad \Leftarrow \quad b = \lambda/2$$

$$= I_0\left(5 + 4\cos \frac{2\pi}{\lambda} n \frac{\lambda}{2}\right) =$$

$$= I_0\left(5 + 4\cos \pi n\right)$$



max: $n = 2, 4, \ldots$

min: $n = 1, 3, \ldots$

Membrane

Write the mode as

$$\Psi(x,y,t) = f(x,y)\cos(\omega t + \phi) \tag{1}$$

and write $f(x,y)$ as a Fourier expansion

$$f(x,y) = \sum_{nm} A_{nm} \sin(k_n x + \phi_n)\sin(k_m y + \phi_m) \tag{2}$$

Apply the boundary conditions (without the fixed point)

$$f(0,y) = f(L,y) = f(x,0) = 0 \tag{3}$$
$$\frac{\partial f}{\partial y}|_{y=L} = 0 \tag{4}$$

This gives

$$k_n = \frac{\pi n}{L} \tag{5}$$
$$k_m = \frac{\pi m - \pi/2}{L} \tag{6}$$
$$\phi_n = \phi_m = 0 \tag{7}$$

so that

$$f(x,y) = \sum_{nm} A_{nm} \sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m - \pi/2}{L}y\right) \tag{8}$$

Substitution into the wave equation gives

$$\omega^2 = v^2\left[\left(\frac{\pi n}{L}\right)^2 + \left(\frac{\pi m - \pi/2}{L}\right)^2\right] \tag{9}$$

A mode of frequency ω thus involves a sum over pairs n and m that satisfy Eq. (9)
Add the fixed point $(x_0, y_0) = (L/2, L)$

$$f(x_0,y_0) = -\sum_{nm} A_{nm} \sin\left(\frac{\pi n}{2}\right)\cos(\pi m) = 0 \tag{10}$$

There are two families of solutions.

The first one is for *even* n's. For even n Eq.(10) is satisfied for arbitrary m, and the mode is

$$\Psi_{nm}(x,y,t) = A_{nm} \sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m - \pi/2}{L}y\right)\cos(\omega t + \phi) \tag{11}$$

The second family is for *odd* n's. In this case the product of the sines is either $+1$ or -1, so this family will consist of pairs, such that one is $+1$ and the other one is -1. Let us take odd n_1, n_2 and arbitrary integers m_1, m_2, and derive the condition for sataisfying Eq.(10):

$$A_{n_1 m_1} \sin\left(\frac{\pi n_1}{2}\right)\cos(\pi m_1) + A_{n_2 m_2} \sin\left(\frac{\pi n_2}{2}\right)\cos(\pi m_2) = 0 \tag{12}$$

which gives for odd n_1, n_2

$$\frac{A_{n_1 m_1}}{A_{n_2 m_2}} = -\frac{\sin\left(\frac{\pi n_2}{2}\right)\cos(\pi m_2)}{\sin\left(\frac{\pi n_1}{2}\right)\cos(\pi m_1)} = -(-1)^{n_2/2 - n_1/2 + m_2 - m_1} \tag{13}$$

Then the mode is

$$\Psi_{n_1 m_1 n_2 m_2}(x,y,t) = \tag{14}$$
$$A_{n_2 m_2}\left[-(-1)^{n_2/2-n_1/2+m_2-m_1}\sin\left(\frac{\pi n_1}{L}x\right)\sin\left(\frac{\pi m_1-\pi/2}{L}y\right)+\sin\left(\frac{\pi n_2}{L}x\right)\sin\left(\frac{\pi m_2-\pi/2}{L}y\right)\right]\cos(\omega t+\phi)$$

The possible values of n_1, n_2, m_1, m_2 follow from the condition (9). In order to know what the possible values of n_1, n_2, m_1, m_2 are, substitute the mode into the wave equation, compare coefficients and obtain:

$$n_1^2 - n_2^2 + (m_1 - 1/2)^2 - (m_2 - 1/2)^2 = 0 \tag{15}$$

So, for example, if $n_1 = n_2$, then $(m_1 - 1/2)^2 = (m_2 - 1/2)^2$; another example $(n1, n2, m1, m2) = (1, 5, 6, 3)$.
In order to find the general solution, one can write an odd n as $n = 2n' - 1$ and get

$$4(n_1' + n_2' - 1)(n_1' - n_2') - (m_1 + m_2 - 1)(m_1 - m_2) = 0 \tag{16}$$
$$4N_1N_2 - M_1M_2 = 0 \tag{17}$$

where the last equation is for new redefined integers $N_1 = n_1' + n_2' - 1, N_2 = n_1' - n_2', M_1 = m_1 + m_2 - 1, M_2 = m_1 - m_2$. So for a given N_1, N_2, M_1, there will be a solution for M_2 if $4N_1N_2/M_1$ is an integer.

$$\Psi(L,t) = C\cos\omega_1 t\cos\omega_2 t$$

הפתרון הכללי למשוואת הגלים:

$$\Psi(x) = \sum_n \left(A_n\cos k_n' x + B_n \sin k_n' x\right)\cos(\omega_n' t + \varphi_n) \qquad \omega = v k$$

תנ״ שפה:

$$\begin{cases} x=0: & T\left.\frac{\partial\Psi}{\partial x}\right|_{x=0} = m\left.\frac{\partial^2\Psi}{\partial t^2}\right|_{x=0} + K\Psi(0) \\ x=L: & \Psi(L,t) = C\cos\omega_1 t\cos\omega_2 t = \frac{1}{2}C\left(\cos((\omega_1-\omega_2)t) + \cos((\omega_1+\omega_2)t)\right) \end{cases}$$

$$\Psi(L,t) = \sum_n \left(A_n\cos k_n' L + B_n\sin k_n' L\right)\cos(\omega_n' t + \varphi_n) =$$
$$= \frac{C}{2}\cos((\omega_1-\omega_2)t) + \frac{C}{2}\cos((\omega_1+\omega_2)t)$$

$$\varphi_n = 0$$

נשווה את האיברים מסכום:

$$A_1\cos k_1' L + B_1\sin k_1' L = \frac{C}{2} \quad ; \quad \omega_1' = \omega_1 - \omega_2$$
$$A_2\cos k_2' L + B_2\sin k_2' L = \frac{C}{2} \quad ; \quad \omega_2' = \omega_1 + \omega_2 \qquad (*)$$

$$\Rightarrow \Psi(x,t) = \left(A_1\cos k_1' x + B_1\sin k_1' x\right)\cos(\omega_1' t) +$$
$$+ \left(A_2\cos k_2' x + B_2\sin k_2' x\right)\cos(\omega_2' t)$$

$$\left.\frac{\partial\Psi}{\partial x}\right|_{x=0} = B_1 k_1'\cos(\omega_1' t) + B_2 k_2'\cos(\omega_2 t)$$

$$\left.\frac{\partial^2\Psi}{\partial t^2}\right|_{x=0} = -\omega_1'^2 A_1\cos\omega_1' t - \omega_2'^2 A_2\cos\omega_2' t$$

$$\Rightarrow \quad T B_1 k_1'\cos\omega_1' t + T B_2 k_2'\cos\omega_2' t =$$
$$= -m\omega_1'^2 A_1\cos\omega_1' t - m\omega_2'^2 A_2\cos\omega_2' t +$$
$$+ K A_1\cos\omega_1' t + K A_2\cos\omega_2' t$$

השוויון נכון לכל t ולכן נשווה את המקדמים של קוסינוסים המתאימים.

$$T B_1 k_1' = -m\omega_1'^2 A_1 + K A_1 \qquad \omega_1' = v k_1' \qquad v = \sqrt{\frac{T}{\rho}}.$$

$$\Rightarrow \begin{cases} T B_1 k_1' = A_1\left(K - m v^2 k_1'^2\right) \\ T B_2 k_2' = A_2\left(K - m v^2 k_2'^2\right) \end{cases}$$

$$\begin{cases} A_1 \cos k_1' L + B_1 \sin k_2' L = \frac{c}{2} \\ A_2 \cos k_2' L + B_2 \sin k_2' L = \frac{c}{2} \end{cases}$$

מתוך (*)

מ-2 המשוואות הנ"ל ניתן למצוא את $A_{1,2}, B_{1,2}$

($\omega'_{1,2}$, $k'_{1,2}$ ידועים מתוך $\begin{cases}\omega_1' = \omega_1 - \omega_2 \\ \omega_2' = \omega_1 + \omega_2\end{cases}$)

$$B_1 = -A_1 \frac{m v^2 k_1'^2 - K}{T k_1'}$$

$$A_1\left(\cos k_1 L - \frac{m v^2 k_1'^2 - K}{T k_1'} \sin k_2' L\right) = \frac{c}{2}$$

$$A_1 = \frac{c/2}{\cos k_1 L - \frac{m v^2 k_1'^2 - K}{T k_1'} \sin k_2' L}$$

$$B_1 = \dots$$

הביטויים ל-A_2, B_2 זהים לביטויים A_1, B_1 אחרי החלפת האינדקסים $2 \Leftrightarrow 1$.

לבסוף:

$$\psi(x,t) = \sum_{i=1}^{2} \left(A_i \cos k_i' x + B_i \sin k_i x\right) \cos(\omega_i' t)$$

$$\omega_1' = \omega_1 - \omega_2 \qquad \omega = v k$$
$$\omega_2' = \omega_1 + \omega_2$$

(5) א) הנחת בוהר: תנועה מעגלית קלאסית של אלקטרון סביב הגרעין והתנע הזוויתי מקוונטל.

כיוון שיש להתחשב בשאלה במסת הגרעין M ומסת האלקטרון m, נגדיר מסה מצומצמת $\mu = \frac{mM}{m+M}$ ונעבור למערכת ייחוס בה המרכז נ"ח.

$$\begin{cases} \frac{kqZq}{r^2} = \frac{\mu v^2}{r} & \text{- חוק ניוטון} \\ L = n\hbar & \text{- הנחת בוהר} \end{cases}$$

$|q|$ - מטען האלקטרון

$Z|q|$ - מטען הגרעין

$$L = \mu v r = n\hbar$$

$$\mu v^2 = \frac{kZq^2}{r} \Rightarrow v^2 = \frac{kZq^2}{\mu r}$$

$$\mu \frac{k^{1/2} Z^{1/2} q}{\mu^{1/2} r^{1/2}} r = n\hbar$$

$$\mu^{1/2} k^{1/2} Z^{1/2} q\, r^{1/2} = n\hbar \Rightarrow \frac{1}{r} = \frac{\mu k Z q^2}{n^2 \hbar^2}$$

$$E_n = \frac{1}{2}\mu v^2 - \frac{kZq^2}{r} = \frac{1}{2}\mu \frac{kZq^2}{\mu r} - \frac{kZq^2}{r} =$$

$$= -\frac{1}{2}\frac{kZq^2}{r} = -\frac{1}{2}\frac{kZq^2 \, \mu k Z q^2}{n^2\hbar^2} =$$

$$= -\frac{\mu Z^2 q^4 k^2}{2\hbar^2}\frac{1}{n^2}$$

מצב היסוד של אטום מימן:

$$q = e,\ Z = 1,\ n = 1$$

$$\mu = \frac{mM}{m+M} = \frac{m}{1 + m/M}$$

$$E_1 = -\frac{m}{1+m/M}\frac{e^4 k^2}{2\hbar^2} = -13.68\,eV$$

$e = 1.6 \cdot 10^{-19}\ C$ $\quad m = 9.1 \cdot 10^{-31}\ kg$ $\quad \hbar = 1.05 \cdot 10^{-34}\ J \cdot s$

$k = 9 \cdot 10^{9}\ \frac{N m^2}{C^2}$ $\quad M = 1.67 \cdot 10^{-27}\ kg$

ב) פוזיטרוניום

$$\mu = \frac{1}{2}m \Leftarrow M = m$$

$$E_1 = -6.85\ eV$$