12. Special issues in perturbation analysis

12.1 Natural scales

In certain problems that have a small nonlinear perturbation, terms of different orders in the small parameter may become comparable in size when the amplitude of the motion reaches a certain magnitude. Formal orders in the expansion lose their significance, and terms of different formal orders compete with each other. When this happens, the ordinary expansion fails, and rescaling of the amplitude may be required in order to define the contributions to various orders of the perturbation in a new manner. This turns out to be the case, for instance, in the analysis of the onset of instabilities in nonlinear systems which undergo soft Hopf bifurcations when a certain parameter crosses a critical value. Analysis of the resulting equation for the amplitude of the nonlinear oscillatory instability (Landau-Stuart or Landau-Ginzburg equations) requires rescaling of the amplitude.

12.1.1 Example

Here we analyze a simple example [25], which provides the relevant insight into the problem. Consider the equation

$$\begin{gathered}\ddot{x} + x + \varepsilon\,\dot{x}^3 + \alpha\varepsilon^2\,\dot{x} = 0 \\ x(0) = 1 \quad , \quad \dot{x}(0) = 0 \quad (\alpha > 0)\end{gathered} \tag{12. 1}$$

The linear part of the equation corresponds to damped harmonic oscillations. The time dependence of the solution of the linear equation is

$$\exp\left[\left(-\tfrac{1}{2}\alpha\,\varepsilon^2 \pm i\,\sqrt{1-\tfrac{1}{4}\alpha^2\varepsilon^4}\right)t\right]$$

Namely, the point (x=dx/dt=0) is an asymptotically stable fixed point of the linear problem: both eigenvalues of the latter have negative real parts. By the Poincaré-Lyapunov theorem (see Chapter 3), the solution of the nonlinear problem must also tend to zero when $t\to\infty$, provided the initial condition is within the Poincaré-Lyapunov domain of the problem (i.e., sufficiently close to the origin). The point (x=dx/dt=0) remains an asymptotically stable fixed point. We will show that this prediction for the exact solution is violated in the present case even by a perturbation method, such as the method of multiple time scales, which is meant to prevent the evolution of secular terms, (that lead to the break down of the expansion at long times). In that method (see Chapter 11) the equations through second order are:

$$D_0^{\,2}x^{(0)} + x^{(0)} = 0 \tag{12. 2a}$$

$$D_0^{\,2}x^{(1)} + x^{(1)} = -2\,D_0D_1x^{(0)} - \left(D_0x^{(0)}\right)^3 \tag{12. 2b}$$

$$\begin{aligned}D_0^{\,2}x^{(2)} + x^{(2)} = &-2\,D_0D_1x^{(1)} - \left(D_1^{\,2} + 2\,D_0D_2\right)x^{(0)} \\ &- 3\left(D_0x^{(0)}\right)^2\left(D_0x^{(1)} + D_1x^{(0)}\right) - \alpha\,D_0x^{(0)}\end{aligned} \tag{12. 2c}$$

The solution of Eq. (12.2a) is

$$x^{(0)} = Ae^{\,i\,T_0} + c.c. \tag{12. 3}$$

where A depends on higher time scales: T_1,T_2,.... With Eq. (12.3), Eq. (12.2b) becomes

$$D_0{}^2 x^{(1)} + x^{(1)} = -2\,i\,D_1 A\, e^{i\,T_0} + i\left(A^3 e^{3\,i\,T_0} - 3A^2\,\overline{A}\,e^{i\,T_0}\right) + c.c. \qquad (12.4)$$

No secular terms appear in the solution if the contribution proportional to $\exp(\pm iT_0)$ vanishes. This implies the "secularity" condition

$$2\,D_1 A + 3\,A^2\,\overline{A} = 0 \qquad (12.5)$$

Writing

$$A = |A| e^{i\,\varphi}$$

we obtain

$$\begin{gathered} D_1 |A|^2 + 3|A|^4 = 0 \\ D_1 \varphi = 0 \end{gathered} \qquad (12.6)$$

which is solved by

$$\begin{gathered} |A| = \frac{1}{\sqrt{3T_1 + C}} \qquad\qquad C = C(T_2, \ldots) \\ \varphi = \varphi(T_2, \ldots) \end{gathered} \qquad (12.7)$$

with C tending to 1 for $t \to 0$ owing to the initial condition imposed in Eq. (12.1). If the present level of accuracy is sufficient, then C=1.

The solution of Eq. (12.2b) now becomes

$$x^{(1)} = -\tfrac{1}{8}\,i\,A^3 e^{3i\,T_0} + B\,e^{i\,T_0} + c.c. \qquad (12.8)$$

With (12.3) for $x^{(0)}$ and (12.8) for $x^{(1)}$, Eq. (12.2c) becomes

$$\begin{aligned} D_0{}^2 x^{(2)} + x^{(2)} = {} & \left(-\tfrac{9}{8}A^5\right) e^{5i\,T_0} + \left(\tfrac{45}{8}A^4\,\overline{A} - 3A^2 D_1 A - 3\,i\,A^2 B\right) e^{3\,i\,T_0} \\ & + \begin{pmatrix} -2\,i\,D_1 B - D_1{}^2 A - 2\,i\,D_2 A - 3A^2 D_1 \overline{A} + 3\,i\,A^2\,\overline{B} \\ +6\,A\overline{A}\,D_1 A + 6\,i\,A\overline{A}\,B + \tfrac{3}{4}A^3\,\overline{A}^2 - i\,\alpha\,A \end{pmatrix} e^{i\,T_0} \\ & + c.c. \end{aligned} \qquad (12.9)$$

Utilizing Eq. (12.5), the secularity condition here becomes

$$D_2 A + \tfrac{15}{4}\,i\,A^3\,\overline{A}^2 + D_1 B - \tfrac{3}{2}A^2\,\overline{B} - 3\,A\overline{A}\,B + \tfrac{1}{2}\alpha\,A = 0 \qquad (12.10)$$

With

$$A = |A| e^{i\,\varphi}$$

Eq. (12.10) yields

$$D_2 |A|^2 + D_1\left(\overline{A}\,B + A\,\overline{B}\right) - 3|A|^2\left(A\,\overline{B} + \overline{A}\,B\right) + \alpha|A|^2 = 0 \qquad (12.11)$$

Inserting Eq. (12.7) for |A|, and writing

$$B = |B| e^{i\psi}$$

Eq. (12.11) gives

$$D_1 Z - \tfrac{9}{2}\frac{Z}{3T_1 + C} + \frac{\alpha}{2(3T_1 + C)^{1/2}} - \frac{D_2 C}{2(3T_1 + C)^{3/2}} = 0 \qquad \left[Z \equiv |B|\cos(\psi - \varphi)\right] \quad (12.12)$$

By Eq. (12.7), C does not depend on T_1. Therefore, Z must be of the form

$$Z = |B|\cos(\psi - \varphi) = (3T_1 + C)^{3/2} V_0 + \tfrac{1}{6}\alpha(3T_1 + C)^{1/2} - \tfrac{1}{12}\frac{D_2 C(T_2, \ldots)}{(3T_1 + C)^{1/2}} \qquad (12.13)$$

with V_0 a function of T_2, T_3,... Thus, the homogeneous contribution in the first order term becomes unbounded in T_1, limiting the validity of the expansion to $t=O(1)$. Hence, the perturbation method fails for long times. The Poincaré - Lyapunov theorem is not obeyed by the approximation.

The origin of this problem is the fact that when

$$x\,,\,\dot{x} \approx \varepsilon^{1/2} \qquad (12.14)$$

the terms $\varepsilon\,\dot{x}^3$ and $\alpha\varepsilon^2\,\dot{x}$ are of similar magnitude. Thus, not always can they be assumed to belong to different orders of expansion. In fact, if we define

$$x = \varepsilon^{1/2}\, y \qquad (12.15)$$

then Eq. (12.1) yield for y the following equation:

$$\ddot{y} + y + \varepsilon^2\left(\dot{y}^3 + \alpha\,\dot{y}\right) = 0$$
$$y(0) = \frac{1}{\varepsilon^{1/2}} \qquad \dot{y}(0) = 0 \qquad (12.16)$$

Both perturbation terms are now of the same order. Defining $\mu=\varepsilon^2$, Eq. (12.16) becomes

$$\ddot{y} + y + \mu\left(\dot{y}^3 + \alpha\,\dot{y}\right) = 0 \qquad (12.16a)$$

The method of multiple time scales now yields ($T_n \equiv \mu^n t$, $D_n \equiv \partial/\partial T_n$):

$$D_0^{\,2} y^{(0)} + y^{(0)} = 0 \quad \Rightarrow \quad y^{(0)} = A' e^{i T_0} + c.c. \qquad (12.17)$$

$$D_0^{\,2} y^{(1)} + y^{(1)} = -2\, D_0 D_1 y^{(0)} - \alpha\, D_0 y^{(0)} - \left(D_0 y^{(0)}\right)^3 =$$
$$\left(-2\,i\, D_1 A' - i\,\alpha A' - 3\,i\, A'^2\,\overline{A'}\right) e^{i T_0} + i\, A'^3 e^{3 i T_0} + \qquad c.c. \qquad (12.18)$$

Eq. (12.18) yields the following secularity condition

$$-2\,i\, D_1 A' - i\,\alpha A' - 3\,i\, A'^2\,\overline{A'} = 0 \qquad (12.19)$$

The equation for the absolute value, $|A'|^2$, is

$$D_1|A'|^2 + \alpha|A'|^2 + 3|A'|^4 = 0 \quad \Rightarrow$$

$$|A'|^2 = \frac{C\,\alpha \exp[-\alpha T_1]}{\alpha + 3C(1 - \exp[-\alpha T_1])} \qquad (T_1 \equiv \mu t = \varepsilon^2 t) \tag{12.20}$$

with C independent of T_1. The initial condition, Eq. (12.16), implies that $C \to 1/\varepsilon$ for $t \to 0$. C may be chosen to be constant at $1/\varepsilon$ at this level of approximation. Thus, for large T_1, $|A'|^2 \to 0$ exponentially. Rescaling enables us to solve the problem through $O(\mu = \varepsilon^2)$, without encountering the problem we have had before rescaling. In the limit $\alpha \equiv 0$ one obtains

$$|A'|^2 \to \frac{C}{1 + 3CT_1} = \frac{C}{1 + 3C\mu t} = \frac{C}{1 + 3C\varepsilon^2 t} = \frac{1}{\varepsilon}\frac{1}{1 + 3\varepsilon t} \tag{12.21}$$

which is the result of the previous treatment (where the contribution of the term proportional to $\alpha\varepsilon^2$ was not included in the given order).

The solution for $y^{(1)}$ is now readily obtained as

$$y^{(1)} = -\tfrac{1}{8}\, i\, A'^3 e^{3iT_0} + B' e^{iT_0} + c.c. \tag{12.22}$$

yielding

$$x = \sqrt{\varepsilon}\left(y^{(0)} + \varepsilon^2 y^{(1)} + \text{"}O(\varepsilon^4)\text{"}\right) \tag{12.23}$$

The last term in Eq. (12.23) has been put in quotation marks. It need not always be $O(\varepsilon^4)$. Nor is the second term always $O(\varepsilon^2)$ relative to the zero-order term. If, for $t=0$, x is $O(1)$ relative to ε, then $y^{(0)}$ is $O(\varepsilon^{-1/2})$. Then, the second term in Eq. (12.23) is $O(\varepsilon^{1/2})$, hence, it is $O(\varepsilon)$ relative to the first term. (Even if one chooses for B' a value that cancels the A'^3 term at $t=0$, soon afterwards they drift apart, so that the statement made above applies). Similarly, the next term is not expected to be $O(\varepsilon^4)$ but $O(\varepsilon^2)$ relative to the zero-order term. Eq. (12.20) implies that, for $t = O(1/\varepsilon^2)$, $|A'|$ becomes $O(1)$, so that x becomes $O(\varepsilon^{1/2})$. Only then do the relative orders of the terms become 1, ε^2, ε^4, respectively, as in the formal expansion. Thus, the expansion is free of secular terms but is not uniform in time. Uniformity is ensured only if $x(0) = O(\varepsilon^{1/2})$ (i.e., small amplitude oscillations). *When this is the case, rescaling accelerates the convergence, as the small parameter in the problem becomes* ε^2.

12.1.2 Natural scales and normal forms

We will now show that the problem encountered in the previous section is a weakness of the method of multiple time scales, but not in the method of normal forms. Consider Eq. (12.1) again. In complex notation ($z = x + iy$, $y = dx/dt$) we have:

$$\dot{z} = -\,i\,z + \varepsilon\underbrace{\tfrac{1}{8}\left(z - z^*\right)^3}_{Z_1} + \varepsilon^2\underbrace{(-)\tfrac{1}{2}\alpha\left(z - z^*\right)}_{Z_2} = 0 \qquad z(0) = 1 \tag{12.24}$$

We expand z in a near identity transformation

$$z = u + \varepsilon T_1 + \cdots \tag{12.25a}$$

and a write normal form

$$\dot{u} = -i\,u + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots \qquad (12.25b)$$

We find in first order:

$$U_1 = [T_1, Z_0] + Z_1 \qquad \Rightarrow$$
$$U_1 = -\tfrac{3}{8} u^2 u^* \qquad (12.26)$$
$$T_1 = \tfrac{1}{16} i\,u^3 + \gamma u^2 u^* - \tfrac{3}{16} i\,u u^{*2} + \tfrac{1}{32} i\,u^{*3}$$

The second order equation yields:

$$U_2 = [T_2, Z_0] + T_1 \frac{\partial Z_1}{\partial u} + T_1^* \frac{\partial Z_1}{\partial u^*} - U_1 \frac{\partial T_1}{\partial u} - U_1^* \frac{\partial T_1}{\partial u^*} + Z_2$$
$$\Rightarrow U_2 = -\tfrac{1}{2}\alpha u + \left(\tfrac{27}{256} i + (\gamma - \gamma^*)\right) u^3 u^{*2} \qquad (12.27)$$

Eqs. (12.26&27) yield the normal form equation through $O(\varepsilon^2)$:

$$\dot{u} = -i\,u - \tfrac{3}{8}\varepsilon u^2 u^* + \varepsilon^2 \left\{ -\tfrac{1}{2}\alpha u + \left(\tfrac{27}{256} i + \tfrac{3}{8}(\gamma - \gamma^*) u^3 u^{*2}\right)\right\} + O(\varepsilon^3) \qquad (12.28)$$

Writing u=ρ·exp(–iθ), the equations for ρ and θ become

$$\dot{\rho} = -\tfrac{3}{8}\varepsilon \rho^3 - \tfrac{1}{2}\alpha \varepsilon^2 \rho + O(\varepsilon^3)$$
$$\theta = t + \varphi \qquad (12.29)$$
$$\dot{\varphi} = \left(-\tfrac{27}{256} + \tfrac{3}{8} i\,(\gamma - \gamma^*)\right)\varepsilon^2 \rho^4 + O(\varepsilon^3)$$

Here γ is a free parameter. Choosing $\gamma - \gamma^* = -(9/64)i$ yields $d\varphi/dt = O(\varepsilon^3)$). We now apply the method of multiple time scales to ρ. The difficulty encountered in Section (12.1) will arise again. The equations for ρ are

$$\rho = \rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + \cdots$$
$$D_0 \rho^{(0)} = 0$$
$$D_0 \rho^{(1)} + D_1 \rho^{(0)} = -\tfrac{3}{8}\left(\rho^{(0)}\right)^3 \qquad (12.30)$$
$$D_0 \rho^{(2)} + D_1 \rho^{(1)} + D_2 \rho^{(0)} = -\tfrac{9}{8}(\rho_0)^2 \rho_1 - \tfrac{1}{2}\alpha \rho^{(0)}$$

The "secularity" conditions here are:

$$D_0 \rho^{(1)} = 0 \qquad\qquad D_0 \rho^{(2)} = 0 \qquad (12.31)$$

yielding the solution:

$$\left(\rho^{(0)}\right)^2 = \frac{R^2}{1 + \tfrac{3}{4} R^2 T_1} \qquad R = R(T_2) \qquad R(0) = 1$$
$$\rho^{(1)} = \left\{ \xi(T_2) - \left(\frac{D_2 R}{R^3} - \tfrac{1}{2}\frac{\alpha}{R} \right) T_1 + \tfrac{3}{4}\alpha R T_1^2 \right\} \left(\rho^{(0)}\right)^3 \qquad (12.32)$$

Thus, a secular behavior occurs. However, it is easy to remedy the situation by solving Eq. (12.29) for ρ directly through order ε^2. The result is

$$\rho^2 = \frac{\frac{4}{3}\alpha\,\varepsilon\rho_0^{\,2}\exp\left[-\alpha\varepsilon^2 t\right]}{\frac{4}{3}\alpha\varepsilon + \rho_0^{\,2}\left(1-\exp\left[-\alpha\varepsilon^2 t\right]\right)} + \begin{cases} O(\varepsilon^2) & t = O(1/\varepsilon) \\ O(\varepsilon) & t = O(1/\varepsilon^2) \end{cases} \tag{12.33}$$

This yields

$$x = w + \varepsilon T_1 + O(\varepsilon^2) \quad , \quad t = O(1/\varepsilon) . \tag{12.34}$$

Since at short times $u=O(1)$, orders of ε in the formal expansion are the actual orders for the period of time relevant. Thus, by employing the method of normal forms, we gain two things. First, we can solve for x explicitly without having to resort to the method of multiple time scales, thereby avoiding the problem encountered earlier. Second, uniformity of the expansion procedure in time is ensured without resorting to rescaling. If the initial condition for x is $O(\varepsilon^{1/2})$ then the *relative errors* rather than the *absolute* ones are as noted in Eqs. (12.33-34).

If, on the other hand, we resale z in Eq. (12.24) by

$$z = \sqrt{\varepsilon}\, w$$

then the equation for w is

$$\dot{w} = -i\, w + \varepsilon^2\left(\tfrac{1}{8}\left(w-w^*\right)^3 - \tfrac{1}{2}\alpha\left(w-w^*\right)\right) \qquad w(0) = \left(1/\sqrt{\varepsilon}\right) \tag{12.35}$$

The lowest term in the normal form is now obtained by a near identity transformation that starts in $O(\varepsilon^2)$ and eliminates all secular terms in this order:

$$\begin{aligned} & w = v + \varepsilon^2 \tilde{T}_2 + \cdots \qquad \Rightarrow \\ & \dot{v} = -i\, v + \varepsilon^2\left(-\tfrac{3}{8}v^2 v^* - \tfrac{1}{2}\alpha v\right) + O(\varepsilon^4) \end{aligned} \tag{12.36}$$

In polar coordinates, $v=\zeta\exp(-i\theta)$, $\theta=t+\varphi$, we obtain

$$\begin{aligned} \dot{\zeta} &= -\varepsilon^2\left(\tfrac{3}{8}\zeta^3 + \tfrac{1}{2}\alpha\zeta\right) + O(\varepsilon^4) \\ \dot{\varphi} &= O(\varepsilon^4) \end{aligned} \tag{12.37}$$

The amplitude ζ can be now solved for either by the method of multiple time scales or directly, yielding the same result:

$$\zeta^2 = \frac{\frac{4}{3}\alpha\zeta_0^{\,2}\exp\left[-\alpha\varepsilon^2 t\right]}{\frac{4}{3}\alpha + \zeta_0^{\,2}\left(1-\exp\left[-\alpha\varepsilon^2 t\right]\right)} + \begin{cases} "O(\varepsilon^2)" & t = O(1/\varepsilon^2) \\ "O(\varepsilon^3)" & t = O(1/\varepsilon) \end{cases} \tag{12.38}$$

with $\zeta_0=\zeta_0(\varepsilon^4 t,...)$. In Eq. (12.38) the apparent errors have been again put in quotation marks. Indeed, since at short times ζ is $O(1/\varepsilon^{1/2})$, the actual *relative* error of the next order term may be lower by one power of ε, as was found out in Section 12.1. However, if one waits for $t=O(1/\varepsilon^2)$, where ζ becomes $O(1)$ (x becomes $O(\varepsilon^{1/2})$), or if the initial condition for x is $O(\varepsilon^{1/2})$ from the start, then the apparent and actual errors coincide. It is worth noting that, when this is the case, the validity of the approximation at a given level of accuracy, say $O(\varepsilon)$, is extended by rescaling. Without rescaling, Eq. (12.34) yields a relative error of $O(\varepsilon^2)$ for $t=O(1/\varepsilon)$ (or $O(\varepsilon)$ for $t=O(1/\varepsilon^2)$), while Eq. (12.38) yields the $O(\varepsilon^2)$ approximation over $t=O(1/\varepsilon^2)$ [$O(\varepsilon)$ for $t=O(1/\varepsilon^3)$]. However, one should bear in mind that this is only correct if the amplitude of the oscillations is small ($O(\varepsilon^{1/2})$). As we shall see, this is typical of the rescaling procedure encountered in the analysis of the onset of nonlinear instabilities. (E. G, the Landau-Stuart and Landau-Ginzburg equations).

Thus, if we restrict ourselves to small oscillations (x starts at $O(\varepsilon^{1/2})$), then rescaling and then analyzing the normal form indicates that the approximation derived is actually valid for a period of time that is longer than proven in a perturbative analysis that does not employ normal forms.

12.1.3 Rescaling - summary of results

12.1.3a A direct application of the method of multiple time scales to Eq. (12.1) leads to the break down of the perturbative expansion in the first order term due to the appearance of secular terms. The latter are a consequence of the existence of a *natural scale* in the problem.

12.1.3b Rescaling the unknown function z, as in Eq. (12.15), yields a problem that can be analyzed by the multiple time scale method without the occurrence of secular terms. However, the resulting expansion is not uniformly valid in time. Although the formal expansion parameter is ε^2, if $x(0)$ is $O(1)$, then the $n+1^{st}$ term in the expansion is $O(\varepsilon)$ relative to the n^{th} term. For $t=O(1/\varepsilon^2)$ x decreases to $O(\varepsilon^{1/2})$, the ratio between consecutive terms becomes $O(\varepsilon^2)$, and the expansion is improved since the *actual* small parameter is then also ε^2.

12.1.3c The normal form equation (Eq. (12.28)) obtained from Eq. (12.1) has the same problem as Eq. (12.1). That is, a multiple time scale analysis yields secular terms that cannot be canceled. However, Eq. (12.28) can be solved directly in the given order, yielding an expansion that is uniform in time, without secular terms. However, the expansion parameter is ε.

12.1.3d Rescaling first and then employing a normal form analysis, yields an equation that can be solved in the given order directly, or by the method of multiple time scales. The results are the same. However, non uniformity of the expansion occurs again in the same manner as in 12.1.3b.

12.1.3.e Rescaling, when required, modifies the formal expansion parameter to a higher order one (ε^2 instead of ε in the example discussed here). However, the actual expansion need not be uniform in time. If $x(0)=O(1)$, then the actual expansion is in powers of ε, not ε^2. Only when x decreases to $O(\varepsilon^{1/2})$, or if it starts there at $t=0$, the actual expansion coincides with the formal one, the expansion parameter becomes ε^2. One order in the expansion is gained.

12.2 More than one small parameter

In some problems, more than one small parameter exists. Sometimes this is a genuine situation, and sometimes it is convenient to introduce more than one small parameter. For example, in Eq. (12.1) one may find it convenient to replace ε^2 by an additional small parameter μ, converting the equation into

$$\begin{gathered} \ddot{x} + x + \varepsilon \dot{x}^3 + \alpha\, \mu\, \dot{x} = 0 \\ x(0) = 1 \qquad \dot{x}(0) = 0 \qquad (\alpha > 0) \end{gathered} \tag{12.39}$$

Exercises

1. a. Perform a double expansion in ε and μ in both the method of multiple time scales and the method of normal forms and find the approximate solution.

   b. What happens in both methods when you make $\mu=\varepsilon^2$ *after* having found the approximate solution?