7. The Poincaré-Lindstedt method

For autonomous conservative systems (e.g., motion in a velocity independent potential), a simple method exists for updating the frequency of motion of the solution. Consider the system

$$\ddot{x} + x = \varepsilon f(x;\varepsilon) \tag{7. 1}$$

We search for the natural time scale of the oscillations. That is, the natural frequency, ω_0 (equals unity here) is to be updated due to the nonlinear perturbation into a new frequency, ω:

$$\omega = 1 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 + \cdots \tag{7. 2}$$

A new time variable is introduced

$$\begin{aligned} &\tau = \omega t \qquad \Rightarrow \frac{d}{dt} = \omega \frac{d}{d\tau} \\ &\frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2} = \left(1 + 2\varepsilon\,\omega_1 + \varepsilon^2\left(\omega_1^{\,2} + 2\omega_2\right) + \cdots\right)\frac{d^2}{d\tau^2} \end{aligned} \tag{7. 3}$$

If we now expand x and $f(x;\varepsilon)$ in powers of ε

$$\begin{aligned} x &= \sum_{n\geq 0} \varepsilon^n\, x_n(\tau) \\ f(x;\varepsilon) &= f_1(x) + \varepsilon\, f_2(x) + \cdots \end{aligned} \tag{7. 4}$$

Inserting the expansions in Eq. (7.1) and comparing coefficients of powers of ε, order by order, we obtain through second order

$$\frac{d^2 x_0}{d\tau^2} + x_0 = 0 \tag{7. 5a}$$

$$\frac{d^2 x_1}{d\tau^2} + x_1 = f_1(x_0) - 2\,\omega_1 \frac{d^2 x_0}{d\tau^2} \tag{7. 5b}$$

$$\frac{d^2 x_2}{d\tau^2} + x_2 = f_2(x_0) + \frac{df_1(x_0)}{dx_0}\, x_1 - 2\,\omega_1 \frac{d^2 x_1}{d\tau^2} - \left(\omega_1^{\,2} + 2\omega_2\right)\frac{d^2 x_0}{d\tau^2} \tag{7. 5c}$$

The lowest order is simply a harmonic oscillation, however, not in ordinary time, but in time measured in units of the modified frequency. The free parameters ω_1, ω_2, ... are selected so that no secular terms develop up to the required order in the perturbation. For instance, in the equation for x_1, the $f_1(x_0)$ may have a simple harmonic term in its Fourier expansion with the same frequency as x_0. ω_1 is chosen so as to eliminate the simple harmonic altogether, since it leads to secular behavior. The solution is obtained as follows

$$x_0 = a\cos(\tau + \varphi_0)$$
$$\frac{d^2 x_1}{d\tau^2} + x_1 = f_1(x_0) + 2\omega_1 a\cos(\tau + \varphi_0) \tag{7. 6}$$

Here φ_0 is a constant phase. We now expand f_1 in a Fourier series

$$f_1(x_0) = \sum_{n\neq 0} a_n \cos n(\tau + \varphi_0) \tag{7. 7}$$

(Note that $f_1(x_0)$ depends only on $\cos(\tau+\varphi_0)$ and not on $\sin(\tau+\varphi_0)$. As a result, its Fourier expansion includes cosines only). Now isolate the Fourier component that resonates with the additional cosine term in Eq. (7.6):

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f_1(a\cos\chi)\cos\chi \, d\chi \tag{7. 8}$$

and require that the two terms cancel each other. That is,

$$\omega_1 = -\tfrac{1}{2}(a_1/a) \tag{7. 9}$$

This can be now carried on to any desired degree of accuracy. It is important to note that the frequency must be updated as we go up in the order of the expansion, since, most probably, the true (but unknown) frequency, ω, is affected by ε to all orders. If we neglect to update the frequency at a given order, then there will be a mismatch between the true frequency, and the approximate one. As a result, the approximate solution will describe the exact one only over a limited range in time: the higher orders will develop secular terms.

The problem exists even in the trivial case of the harmonic oscillator with a slightly perturbed frequency of Section 5

$$\ddot{x} + (1 + 2\varepsilon)x = 0 \tag{7.10}$$

We, of course, know how to solve this linear problem, but pretend that it requires a perturbative treatment. Here $f(x;\varepsilon)$ of Eq. (7.1) is simply

$$f(x;\varepsilon) = f_1(x) = -2x \tag{7.11}$$

Inserting Eq. (7.11) in Eq. (7.8) we find that a_1=-2a. Hence, Eq. (7.9) yields

$$\omega_1 = 1 \tag{7.12}$$

When inserted in Eq. (7.5b), this yields

$$x_1 = b\cos(\tau + \psi_0) \tag{7.13}$$

Here, again, ψ_0 is a constant phase. Eq. (7.5c) now becomes

$$\frac{d^2x_2}{dt^2} + x_2 = (1 + 2\omega_2)a\cos(\tau + \varphi_0) \tag{7.14}$$

In order not to have secular terms here, we must have

$$\omega_2 = -\frac{1}{2} \tag{7.15}$$

yielding

$$x_2 = c\cos(\tau + \xi_0) \tag{7.16}$$

with ξ_0 a constant phase. As a result, through second order,

$$\begin{aligned} &\tau = \left(1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)\right)t \\ &x = a\cos(\tau + \varphi_0) + \varepsilon b\cos(\tau + \psi_0) + c\cos(\tau + \xi_0) + O(\varepsilon^3) \qquad t = O(\varepsilon^0) \end{aligned} \tag{7.17}$$

An $O(\varepsilon^2)$ error is obtained for $t \le O(1/\varepsilon)$ if the ε^2 term is dropped. Now, as an example, impose the initial condition $x(0)=1$, $dx/dt(0)=0$. [Remember that $dx/dt = \omega(dx/dt)$!]. This yields $a=1$, $b=c=0$, $\varphi_0=0$, giving

$$x = \cos\tau = \cos\left(1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)\right)t$$

which, for $t \le O(1/\varepsilon)$, yields:

$$x = \cos\left(1 + \varepsilon + \tfrac{1}{2}\varepsilon^2\right)t + O(\varepsilon^2) \qquad t \le O(1/\varepsilon) \tag{7.18}$$

It is easy to see that this is simply the beginning of the expansion of the exact known solution

$$x = \cos\sqrt{1 + 2\varepsilon}\,t$$

and that, indeed the error incurred is $O(\varepsilon^2)$ for $t \le O(1/\varepsilon)$.

As another example, consider, again, the Duffing oscillator:

$$\ddot{x} + x = -\varepsilon x^3 \qquad x(0) = 1, \quad \dot{x}(0) = 0 \tag{7.19}$$

Here

$$f(x;\varepsilon) = f_1(x) = -x^3 \tag{7.20}$$

and

$$f_1(x_0) = -a^3\cos^3(\tau+\varphi_0) = -\tfrac{1}{4}a^3\left(\cos 3(\tau+\varphi_0) + 3\cos(\tau+\varphi_0)\right) \tag{7.21}$$

The coefficient of the resonant term is

$$a_1 = -\tfrac{3}{4}a^3 \tag{7.22}$$

yielding

$$\omega_1 = \tfrac{3}{8}a^2 \tag{7.23}$$

This is precisely the $O(\varepsilon)$ correction to the frequency found in Section 6.5 (Eq. (6.51)). Higher orders may be computed in a similar manner.

Finally, we point out why the method fails when the function f of Eq. (7.1) depends on the velocity, dx/dt. Consider the lowest order contribution, f_1. Regardless of whether the solution tends to a periodic limit cycle, or is not be periodic, $f_1(x_0)$ is now replaced by $f_1(x_0, dx_0/d\tau)$, with

$$f_1\left(x_0, \frac{dx_0}{d\tau}\right) = f_1\left(a\cos(\tau+\varphi_0), -a\sin(\tau+\varphi_0)\right)$$

The corresponding Fourier expansion now is

$$f\left(x_0, \frac{dx_0}{d\tau}\right) = \sum_{n\geq 0}\left\{a_n\cos n(\tau+\varphi_0) + b_n\sin n(\tau+\varphi_0)\right\}$$

Now, in Eq. (7.6), f_1 will contribute two resonant terms: a cosine term which can be eliminated by an appropriate choice of ω_1 and a sine term that one cannot eliminate. The latter will produce a secular term. Hence the method works only for $t=O(1)$.

Exercises

7.1 Prove Eqs. (7.5a-c).
7.2 Use the Poincaré-Lindstedt method to find the first order correction to the frequency in

$$\ddot{x} + x + \varepsilon\sin x = 0$$