5. Secular terms

The validity of *naive time-dependent perturbation theory* is often limited to short periods of time. The best way to see this is through a few examples. The example of the precession of Mercury, discussed in Section (4.4) (see Eq. (4.40) and the discussion following it) has been one such example.

As a further one, consider the linear problem of Section 4.3 (Harmonic oscillator with slightly modified frequency):

$$\ddot{x} + (1 + 2\varepsilon)x = 0 \qquad\qquad x(0) = 1,\ \dot{x}(0) = 0 \tag{5. 1}$$

Expand $x(t)$ in powers of ε

$$x(t) = \sum_{n=0}^{\infty} \varepsilon^n x_n(t) \tag{5. 2}$$

Inserting Eq. (5.2) in Eq. (5.1) and comparing terms order by order in powers of ε, we obtain

$$\begin{aligned} &\ddot{x}_0 + x_0 = 0 \\ n \geq 1 \qquad &\ddot{x}_n + x_n + 2x_{n-1} = 0 \\ \Rightarrow x_0 = \cos t \quad x_1 = -t\sin t \quad &x_2 = \tfrac{1}{2}t^2\cos t - \tfrac{1}{2}t\sin t \qquad \cdots\cdots \end{aligned} \tag{5. 3}$$

Thus, a naive expansion around a zero-order term, with a frequency identical to the unperturbed one, is valid only for $t=O(\varepsilon^0)$. The reason is that frequency of the correct solution is modified by ε.

This can be also seen this is by solving Eq. (5.1) using the Green's function of the unperturbed harmonic oscillator, including the initial condition):

$$x = \cos t - 2\varepsilon \int_0^t \sin(t-s)x(s)\,ds \tag{5. 4}$$

(Check that the solution, Eq. (5.4), satisfies Eq. (5.1).) Iterating Eq. (5.4) for $x(s)$ we obtain

$$\begin{aligned} x &= \cos t - 2\varepsilon \int_0^t \sin(t-s)\left(\cos s - 2\varepsilon \int_0^s \sin(s-s')x(s')\,ds'\right)ds \\ &= \cos t - \varepsilon t\sin t + 4\varepsilon^2 \int_0^t \sin(t-s)\,ds \int_0^s \sin(s-s')x(s')\,ds' \end{aligned} \tag{5. 5}$$

Use of the unperturbed Green's function yields the same expansion, valid only for $t=O(\varepsilon^0)$.

Next consider the equation

$$\dot{x} = \left(1 + \varepsilon + \varepsilon^2\right)x \qquad x(0) = 1 \tag{5. 6}$$

It is trivially solved by

$$x = \exp\left[\left(1 + \varepsilon + \varepsilon^2\right)t\right] \tag{5. 7}$$

However, a naive expansion of x in powers of ε yields

$$x = 1 + (\varepsilon t) + \varepsilon^2\left(t + \tfrac{1}{2}t^2\right) + \cdots \tag{5. 8}$$

This expansion is valid for all times, because we are lucky. The expansion of the exponential function converges for all values of the argument. However, the expansion is not valid *uniformly* in time. If we want to approximate the exact solution with an error of $O(\varepsilon)$, the number of terms we have to include changes with the time span for which we want the approximation to hold. For $t \le O(\varepsilon^0)$, it is sufficient to keep the zero-order term only. For $t \le O(1/\varepsilon)$, on the other hand, *all terms* of the form $[(\varepsilon t)^n/(n!)]$ must be kept, because they become $O(\varepsilon^0)$ when t becomes $O_S(1/\varepsilon)$. Thus,

$$x = \begin{cases} 1 + O(\varepsilon) & t \le O\left(\varepsilon^0\right) \\ \\ \exp\left[(1+\varepsilon)t\right]\left\{1 + O(\varepsilon)\right\} & t \le O(1/\varepsilon) \end{cases} \tag{5. 9}$$

The validity of the approximation depends on the time span.

In linear problems this obstacle is not an issue as we have ways of overcoming it. In nonlinear problems it is a major issue, requiring the development of special expansion methods that are "uniformly" valid in time. That is, the expansion is valid for all times, and, in practice, for relatively long times (e.g., $t \le O(1/\varepsilon)$ instead of $t \le O(\varepsilon^0)$ only).

<u>Comment</u> The usual time-dependent perturbation theory in quantum mechanics is of the "naive" type. One may therefore ask oneself why is it that Fermi's Golden Rule for transition probabilities works so well. The answer is that the exponential decay of the wave-function is what saves us. Indeed, the perturbation method is valid only for short times, but we don't have to worry about long times thanks to the exponential decay.