3. Stability of solutions

Often, one is interested in the stability of solutions of the dynamical equations. For example, if a perturbation method is employed, it is desirable to know for how long in time the approximation is be close to the exact, but unknown, solution. In other problems one is interested in finding whether the solution of a given problem does or does not approach some asymptotic form. If it does, the rate of approach may be of interest.

Let $\phi(t)$ be a solution of the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}; \varepsilon) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3. 1}$$

3.1 Liapounov stability

The first question that arises is the degree of sensitivity of the solution to a slight change in the initial condition. Let $\varepsilon > 0$ be an allowed deviation in the solution, and $\delta > 0$ - the allowable range of variation in the initial condition. If, for any arbitrarily small ε, there exists a δ such that when the initial condition $\mathbf{x}_0$ is replaced by $\mathbf{x}_0'$ satisfying

$$\left\| \mathbf{x}_0 - \mathbf{x}_0' \right\| \le \delta \tag{3. 2}$$

the solution $\xi(t)$ emanating from the new initial condition satisfies

$$\left\| \xi(t) - \phi(t) \right\| < \varepsilon \tag{3. 3}$$

$\phi(t)$ is called stable in the Liapounov sense. This is shown in Fig. 3.1.

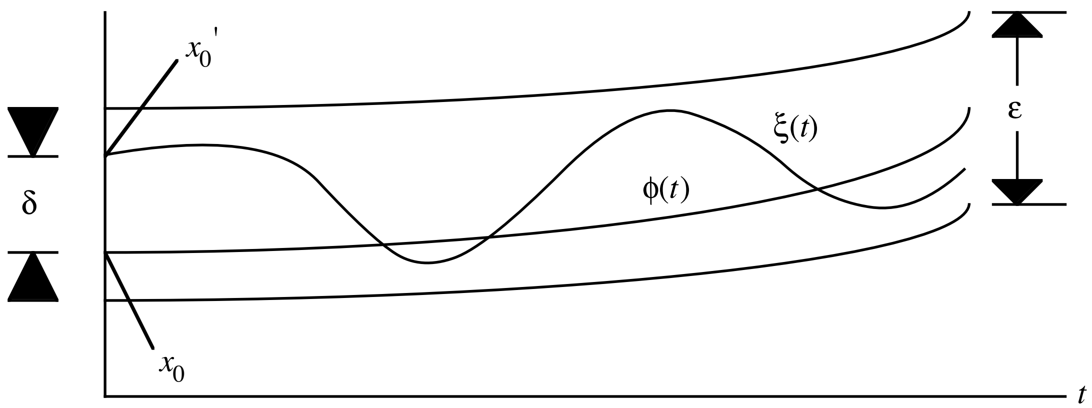
<u>Example:</u> a harmonic oscillator

Consider

$$\frac{d^2 x}{dt^2} = -x \tag{3. 4}$$

which can be transformed into two first order equations:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{3. 5}$$



<u>Fig. 3.1</u>

Define

$$z \equiv \begin{pmatrix} x \\ y \end{pmatrix}$$

then z=(0,0) is clearly a solution. Assume that the initial condition for the problem, $z_0=(x_0,y_0)$ satisfies

$$\begin{aligned} \|z_0\| = |x_0| + |y_0| \le \delta \\ \Rightarrow |x_0| \le \delta \qquad\qquad |y_0| \le \delta \end{aligned} \tag{3. 6}$$

This equation is solved by

$$\begin{aligned} z = \begin{pmatrix} x \\ y \end{pmatrix} = \exp\left(-\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \\ \left(\cos t - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin t\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ y_0 \cos t + x_0 \sin t \end{pmatrix} \end{aligned} \tag{3. 7}$$

which satisfies

$$\|z\| = |x| + |y| \le 2(|x_0| + |y_0|) \le 2\delta \tag{3. 8}$$

Thus, it is sufficient to choose δ=(ε/2) in order to satisfy the Liapounov stability condition. The intuitive meaning is that if one chooses an initial condition which is within a certain circle of a small radius from z=0, then the solution always stays within the same circle. However, it does not tend to zero necessarily.

3.2 Positive attractor

The solution φ(t) is a positive attractor if a positive constant δ exists such that

$$\left\|\mathbf{x}_0{}' - \mathbf{x}_0\right\| \le \delta \Rightarrow \lim_{t\to\infty}\|\mathbf{x}(t) - \phi(t)\| = 0 \tag{3. 9}$$

That is, $\mathbf{x}(t)$ tends towards $\phi(t)$, but needs not be always in a δ-neighborhood of $\phi(t)$.

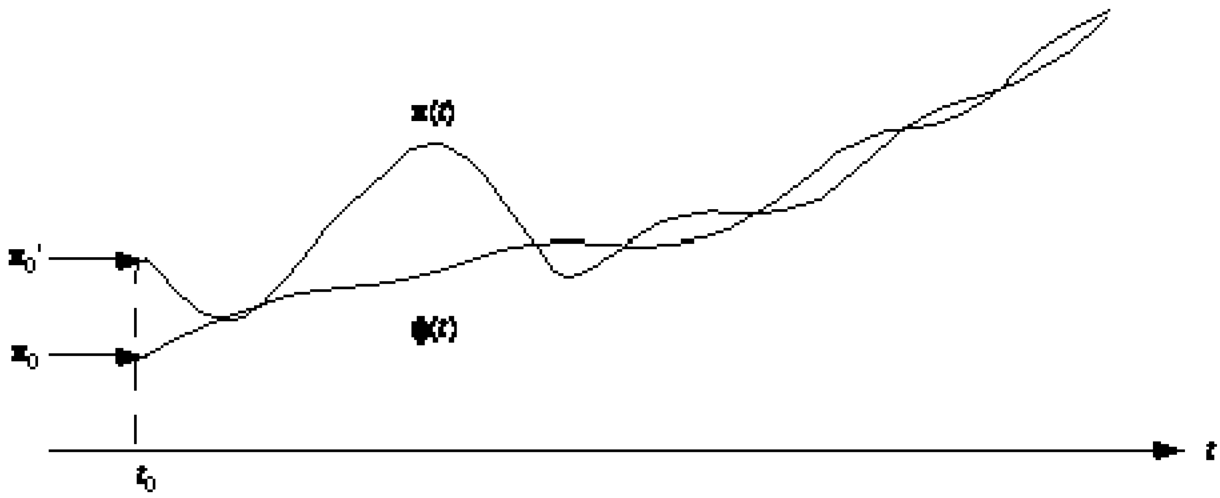


Fig. 3.2

3.3 Asymptotic stability

$\phi(t)$ is called asymptotically stable if it is a Liapounov stable positive attractor. Then, a solution $\mathbf{x}'(t)$ which starts close to $\phi(t)$, at t=0 (i.e., within a δ neighborhood) not only stays close to the latter for all times, but tends to the latter for $t \to \infty$.

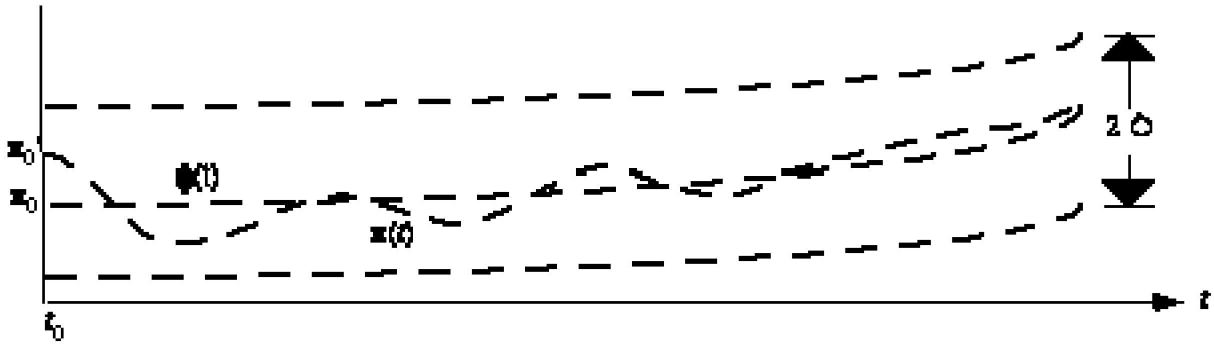


Fig. 3.3

3.4 Examples(two curves in the plane)

Liapounov stability. The initial conditions are within δ of each other, the solutions remain within distance ε, but do not approach each other asymptotically:

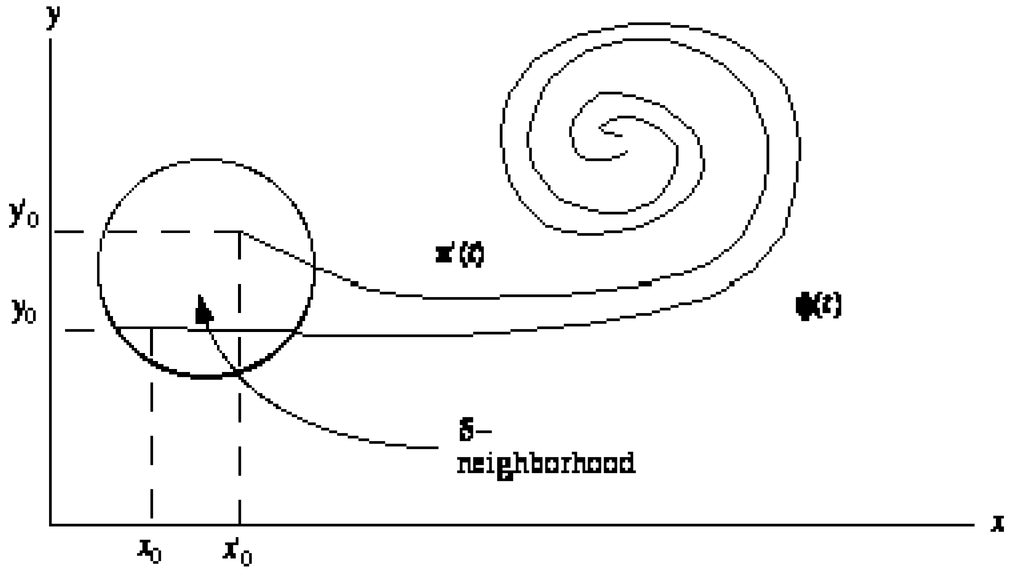


Fig. 3.4

Positive attractor. Solutions start at a distance ≤ δ and approach each other as *t* tends to infinity although not always are they within ε of each other.

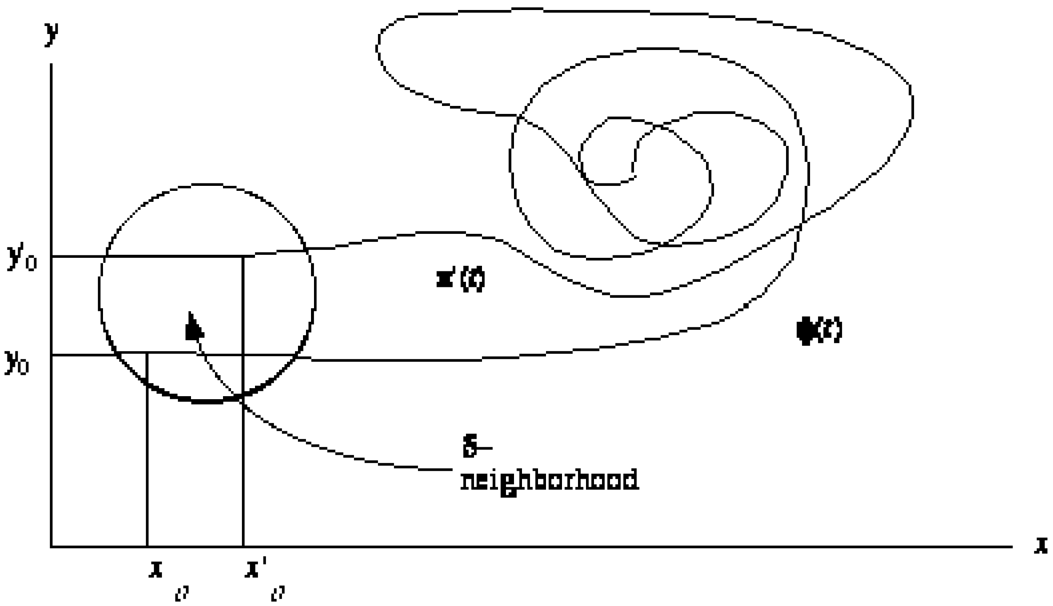


Fig. 3.5

Asymptotic stability. Solutions are always close to each other and approach one another as t tends to infinity.

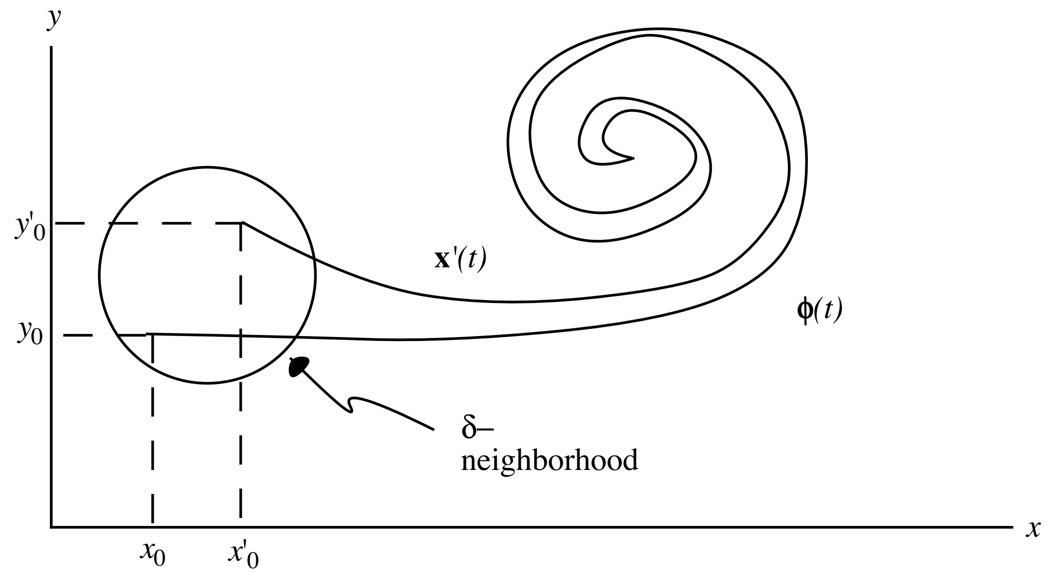


Fig. 3.6

3.5 Poincaré - Liapounov theorem

The Poincaré - Liapounov theorem is an important tool in the analysis of the stability of nonlinear systems. We follow the presentation in [8].

Consider the initial value problem

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}(t))\mathbf{x} + \mathbf{g}(t,\mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \qquad t \geq t_0 \qquad \mathbf{x}, \mathbf{x}_0 \in \mathrm{R}^{\mathrm{n}} \tag{3.10}$$

where **A** is an n×n constant matrix. All the eigenvalues of **A** have *negative real parts*. **B**(t) is an n×n matrix, continuous in t and satisfying

$$\lim_{t\to\infty} \|\mathbf{B}(t)\| = 0 \tag{3.11}$$

The vector field **g**(t,**x**) is continuous in t and in **x**, has a continuous derivative in **x**, and satisfies

$$\mathbf{g}(t,\mathbf{x}) = o(\|\mathbf{x}\|) \\ \|\mathbf{x}\| \to 0 \tag{3.12}$$

uniformly in t. (That is, when $\|\mathbf{x}\|\to 0$, $(\|\mathbf{g}(t,\mathbf{x})\|/\|\mathbf{x}\|\to 0)$ at a rate which is independent of t).

Under these conditions, three constants C,μ,and δ exist such that

$$\|\mathbf{x}_0\| < (\delta/C) \quad \Rightarrow \quad \|\mathbf{x}(t)\| < C\|\mathbf{x}_0\|\exp(-\mu(t - t_0)) \tag{3.13}$$

Significance of theorem The nonlinear perturbation, **g**(t,**x**), is "small" compared to **x** when the latter tends to zero, so that it cannot destroy the stability of the linear problem (i.e., Eq. (3.10) with **g** omitted).

Comment The range $\|\mathbf{x}_0\| \leq (\delta/C)$, within which the attraction of the solution to zero is exponential, is called the Poincaré - Liapounov domain of the equation.

Comment The existence and uniqueness theorem stated in Section 2.3 guarantees that a unique solution exists for some time interval $[t_0,t_1]$. The theorem states that the solution exists for all t and tends to zero exponentially when $t\to\infty$.

Proof

Define

$$\mathbf{x}(t) = \exp(\mathbf{A}(t - t_0))\mathbf{y}(t) \tag{3.14}$$

This gives

$$\dot{\mathbf{x}} = \mathbf{A}\exp(\mathbf{A}(t - t_0))\mathbf{y} + \exp(\mathbf{A}(t - t_0))\dot{\mathbf{y}} \tag{3.15}$$

Inserting in Eq. (3.10) we find

$$\dot{\mathbf{y}} = \exp(-\mathbf{A}(t - t_0))(\mathbf{B}(t)\mathbf{x} + \mathbf{g}(t,\mathbf{x})) \tag{3.16}$$

Solving Eq. (3.16) for y, we find for x

$$\mathbf{x}(t) = \exp\big(\mathbf{A}(t-t_0)\big)\mathbf{x}_0 + \int_{t_0}^{t} \exp\big(\mathbf{A}(t-s)\big)\big(\mathbf{B}(s)\mathbf{x} + \mathbf{g}(s,\mathbf{x}(s))\big)ds \tag{3.17}$$

<u>Claim</u> Since all eigenvalues of **A** have negative real parts, constants $C>0$ and $\mu_0>0$ exist, such that

$$\left\| e^{A(t-t_0)} \right\| < C\, e^{-\mu_0 (t-t_0)}. \tag{3.18}$$

For the sake of simplicity, let us assume that **A** is diagonalizable. (The proof when A is not diagonalizable is more cumbersome, but rather similar). There, therefore, exists a nonsingular matrix P such that

$$\mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \ddots & 0 & \\ & 0 & \ddots & \\ & & & \lambda_n \end{pmatrix} \tag{3.19}$$

As a result,

$$\begin{aligned} &\exp\big(\mathbf{A}(t-t_0)\big) = \mathbf{P}\exp\big(\mathbf{\Lambda}(t-t_0)\big)\mathbf{P}^{-1} \;\Rightarrow \\ &\big\|\exp\big(\mathbf{A}(t-t_0)\big)\big\| = \sum_{ij}\Big|\big(\exp\big(\mathbf{A}(t-t_0)\big)\big)_{i,j}\Big| = \\ &\sum_{ij,k}\Big|\mathrm{P}_{ik}\exp\big(\lambda_k(t-t_0)\big)\mathrm{P}^{-1}_{k,j}\Big| \le \sum_k \exp\big(-\mathrm{Re}\lambda_k(t-t_0)\big)\sum_{i,j}\big|\mathrm{P}_{ik}\big|\big|\mathrm{P}^{-1}_{k,j}\big| \end{aligned} \tag{3.20}$$

Choose

$$\mu_0 = \inf_{1\le k\le n}\left|\mathrm{Re}\,\lambda_k\right| \tag{3.21}$$

to obtain

$$\big\|\exp\big(\mathbf{A}(t-t_0)\big)\big\| < \exp\big(-\mu_0(t-t_0)\big)C \qquad C = \sum_{i,j}\big|\mathrm{P}_{ik}\big|\big|\mathrm{P}^{-1}_{k,j}\big| \tag{3.22}$$

In addition, since we have

$$\mathbf{B}(t)\xrightarrow[t\to 0]{} 0, \qquad \frac{\|\mathbf{g}\|(t,\mathbf{x})}{\|\mathbf{x}\|}\xrightarrow[t\to 0]{} 0 \tag{3.23}$$

there will always be an $\eta(\delta) > 0$ such that

$$\|\mathbf{x}\| \le \delta \Rightarrow \begin{cases} \|\mathbf{B}(t)\| \le \eta(\delta), & t \ge t_0 \\ \|\mathbf{g}(t,\mathbf{x})\| \le \eta(\delta)\|\mathbf{x}\| & \end{cases} \tag{3.24}$$

As a result, as long as $\|\mathbf{x}\|\le\delta$, we have

$$\|\mathbf{x}(t)\| \le \|\exp(\mathbf{A}(t-t_0))\|\|\mathbf{x}_0\| + \int_{t_0}^{t} \|\exp(\mathbf{A}(t-s_0))\|\{\|\mathbf{B}(s)\|\|\mathbf{x}(s)\| + \|\mathbf{g}(s,\mathbf{x}(s))\|\}\,ds \le$$
$$C\exp(-\mu_0(t-t_0))\|\mathbf{x}_0\| + \int_{t_0}^{t} C\exp(-\mu_0(t-t_0))\,2\,\eta(\delta)\|\mathbf{x}(s)\|\,ds \tag{3.25}$$

Multiply both sides of Eq. (2.63) by $\exp[\mu_0(t-t_0)]$ and obtain

$$\underbrace{\exp(\mu_0(t-t_0))\|\mathbf{x}(t)\|}_{|z|} \le \underbrace{C\|\mathbf{x}_0\|}_{\delta_3} + \underbrace{2\eta(\delta)C}_{\delta_1}\int_{t_0}^{t} \exp(\mu_0(t-s))\|\mathbf{x}(s)\|\,ds \tag{3.26}$$

This special case of the Gronwall lemma (Eq. (2.35)) with $\delta_2=0$, yields

$$\exp(\mu_0(t-t_0))\|\mathbf{x}(t)\| \le C\|\mathbf{x}_0\|\exp\{2\eta(\delta)C(t-t_0)\} \quad \Rightarrow$$

$$\|\mathbf{x}(t)\| \le C\|\mathbf{x}_0\|\exp\{-(\mu_0 - 2\eta(\delta)C)(t-t_0)\} = \tag{3.27}$$

$$C\|\mathbf{x}_0\|\exp(-\mu(t-t_0)), \qquad \mu = \mu_0 - 2\eta(\delta)C$$

Now choose $\mathbf{x}_0$ so that

$$\left\|\mathbf{x}_0 \le \min\left(\delta, \frac{\delta}{C}\right)\right\| \tag{3.28}$$

Continuity of the solution of the differential equation guarantees that there is a range $[t_0,t_1]$ within which $\|\mathbf{x}(t)\|\le\delta$. Therefore, in this interval $\mathbf{x}(t)$ will fall off exponentially (see Eq. (3.27)). Now, consider t_1 as a starting point for solving Eq. (3.10) with

$$\|\mathbf{x}(t_1)\| \le \delta \tag{3.29}$$

as an initial condition $\mathbf{x}(t_1)$. A solution, bounded by a falling exponential will be found in an additional interval $[t_1,t_2]$. In the latter, again, $\|\mathbf{x}(t)\|\le\delta$. Now, the next interval, $[t_2,t_3]$ can be found, and so on. It is not difficult to show that in this way all $t_0 \le t < \infty$ are covered.

Why are we interested in the vicinity of zero?
For any other equation, with a solution $\mathbf{x} = \phi(t)$, the asymptotic stability or instability of $\phi(t)$ is analyzed by studying the behavior of

$$\mathbf{y}(t) = \mathbf{x}(t) - \phi(t) \tag{3.30}$$

around zero.

3.5.1 Difference between nonlinear and linear problems

This theorem points out the difference between linear and nonlinear problems. Asymptotic stability of the solution in the linear problem does not guarantee asymptotic stability in the nonlinear one, except when the added nonlinearity is small. In fact, in linear problems, sometimes the addition of a small perturbation to an equation that originally had an asymptotically stable solution, may destroy this stability [9]. As an example, consider a two-dimensional problem

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{g}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} 0 & 0 \\ a^2\varepsilon^2 & 0 \end{pmatrix}\mathbf{x} \tag{3.31}$$

$$\Rightarrow \|\mathbf{A}\| = 1 + 2\varepsilon \qquad \|\mathbf{g}\| = a^2\varepsilon^2$$

Thus, a small (O(ε^2)) linear perturbation is added. **A** has one (double) eigenvalue: $-\varepsilon$, while the eigenvalues of the matrix of the full equation,

$$\dot{\mathbf{x}} = \begin{pmatrix} -\varepsilon & 1 \\ a^2\varepsilon^2 & -\varepsilon \end{pmatrix}\mathbf{x} \tag{3.32}$$

are

$$-\varepsilon \pm a\varepsilon$$

Thus, the stability properties of the problem change completely when a small perturbation is added:

a < 1	stable
a > 1	unstable .

The reason is that although the perturbation is of higher order in the small parameter, ε, it does not vanish faster than **x** when $\mathbf{x} \to 0$.

3.5.2 One dimensional examples to Poincaré - Liapounov theorem

I.

$$\frac{dx}{dt} = -\frac{x}{1+x} \tag{3.33}$$

This equation is solved by

$$\frac{x}{x_0}\exp(x - x_0) = \exp(-(t - t_0)) \tag{3.34}$$

Clearly, for any initial condition $x_0 > -1$,

$$x \xrightarrow[t\to\infty]{} 0 \tag{3.35}$$

Let us study this problem for $|x| \ll 1$. The right hand side of Eq. (3.33) can be written as

$$\frac{dx}{dt} = -x + \frac{x^2}{1+x} \tag{3.36}$$

The nonlinear perturbation vanishes faster than the linear part as $x \to 0$. Hence, the conditions of the Poincaré - Liapounov theorem are obeyed, and the stability properties of the solution of the linear problem

$$\frac{dx}{dt} = -x, \qquad x = x_0\exp(-(t - t_0)) \tag{3.37}$$

is carried over to the nonlinear one, and x=0 remains an asymptotically stable attractor of as $t \to \infty$.

II.

$$\frac{dx}{dt} = -1 + \exp(-x) \tag{3.38}$$

Clearly, x≡0 is a solution. Is it an (asymptotically stable) attractor? Let us single out the linear part:

$$\frac{dx}{dt} = -x + \underbrace{\{\exp(-x) - (1-x)\}}_{\text{nonlinear perturbation}} \tag{3.39}$$

Clearly, in its present form the equation satisfies the conditions of the theorem, so that for |x|<1 zero is an attractor. A slightly different analysis yields similar results:

$$\begin{aligned} & y = \exp(-x) \quad \Rightarrow \quad \frac{dy}{dt} = -y(y-1) \\ & \left(\frac{y-1}{y_0-1}\right)\frac{y_0}{y} = \exp(-(t-t_0)) \quad \Rightarrow \quad y = \frac{y_0}{\left[y_0 - (y_0-1)\exp(-(t-t_0))\right]} \end{aligned} \tag{3.40}$$

y≡1 is a solution of Eq. (3.40). Is it stable? Write

$$y = 1 + \xi \tag{3.41}$$

This yields

$$\frac{d\xi}{dt} = -\xi + \xi^2 \tag{3.42}$$

for which ξ=0 remains an attractor (the nonlinear part vanishes more rapidly when x≡0, so that the conditions for the Poincaré - Liapounov theorem are satisfied) when $t \to \infty$. The same is therefore true for y=1 and, consequently, for x=0.

Exercises

3.1 Show that x=0 is a Liapounov stable solution of the equation

$$\frac{dx}{dt} = -k\,x \qquad x(0) = 1$$

3.2 Show that for a spring with friction:

$$\frac{d^2x}{dt} + \gamma\frac{dx}{dt} + x = 0, \qquad \gamma > 0$$

the point x=0, dx/dt=0 is an asymptotically stable solution.

3.3 Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two different solutions of Eq. (3.10). Show that

$$\|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq C\|\mathbf{x}_1(t_0) - \mathbf{x}_2(t_0)\|\exp(-\mu(t - t_0))$$

that is, although the two solutions start at different points, each of them converges to zero asymptotically, and so does the distance between them.

3.4 Show that any solution of the equation

$$\dot{x} = x, \quad x(0) = x_0$$

is unbounded and unstable in the Liapounov sense.

3.5 Show that any solution of the equation

$$\dot{x} = 1, \quad x(0) = x_0$$

is unbounded, but is stable in the Liapounov sense.