

INTRODUCTION TO NONLINEAR DYNAMICS

(Lecture notes, 1998/99)

PART I: INTRODUCTION

1. Nonlinear phenomena - what's all this fuss about?

Nonlinear phenomena are encountered in all areas of the quantitative sciences. Today it is becoming clearer that the linear world, for which mathematics has been developed over a period of 300 years is but a tiny corner of a far richer world which is being unraveled slowly.

A few examples are:

nonlinear mechanical vibrations,
 population dynamics (e.g., evolution of predator-prey populations),
 electronic circuits (e.g., triodes),
 laser physics,
 astrophysics (e.g., planetary motion),
 the heart-beat,
 biological clocks,
 war games,
 economical dynamics(?????????),
 nonlinear diffusion:
 fluid dynamics,
 meteorology,
 plasma physics,
 chemical reactions in solutions,
 Stefan problem (diffusion in domains with moving boundaries),
 nonlinear wave equations,
 time-delay process,
 critical phenomena (phase transitions).

1.1 Some history

Already in the 18th century, Clairot, Lagrange and Laplace developed perturbation methods for the treatment of nonlinear problems in celestial mechanics. Jacobi, Poincaré and Lindstedt studied similar problems towards the end of the 19th century. Quite a few concepts that have evolved in recent years can be traced back to the work of Poincaré. Towards the end of the 19th century Rayleigh studied self sustained vibrations in his work on the theory of sound. His equation for the self sustaining oscillations in organ pipes,

$$\ddot{u} + u = \varepsilon \left(\dot{u} - \frac{1}{3} \dot{u}^3 \right) \quad (1.1)$$

is equivalent to the equation developed by Van der Pol in the 1920's for the current in a triode,

$$\ddot{u} + u = \varepsilon (1 - u^2) \dot{u} \quad (1.2)$$

Both equations yield anharmonic self-sustaining oscillations.

During the 1930's to 1940's Krylov and Bogoliubov developed methods for analyzing small nonlinear perturbations. Later on, interest arose in nonlinear problems in which the nonlinearity is not small. Important milestones were (among others) the Lorenz equations as a model for fluid motion, Feigenbaum's analysis of discrete nonlinear maps and their relevance to the behavior of

continuous nonlinear systems, and the advent of the concepts of chaos and fractals.

1.2 Some inherent features

1.2.1 Time scales

Often, one is interested in harmonic oscillations that are perturbed by a small nonlinearity. The motion may display stable (almost harmonic) or unstable (damped or growing amplitude) oscillations. In either case, the time scale over which significant changes occur in the system (period of oscillations in the oscillatory case, or time scale for amplitude damping or growth) is affected by the nonlinearity: it depends on the amplitude of the oscillations. This is unlike the case of harmonic motion, where the period of oscillations is independent of the amplitude. In truth, this should not surprise us, as we have already encountered in our earlier education oscillatory systems in which the amplitude and the period are related. In particular, remember Kepler's law for planetary motion. If we denote the period of motion of the planet around the sun by T , and the amplitude of the orbit by x_0 , we have

$$T^2 \propto x_0^3 \quad (1.3)$$

The general case for conservative systems is well demonstrated by the case of homogeneous potentials [1]. The equation of motion of a point particle in a one-dimensional potential is

$$m\ddot{x} + \frac{\partial V}{\partial x} = 0 \quad (1.4)$$

The total energy, E , is given by

$$E = \frac{1}{2} m \dot{x}^2 + V(x) \quad (1.5)$$

Solving for dx/dt , one has for the period of the closed orbit motion

$$x = \pm \sqrt{\frac{2(E - V(x))}{m}} \Rightarrow T = \oint \sqrt{\frac{m}{2(E - V(x))}} dx \quad (1.6)$$

For a symmetric homogeneous potential

$$V(x) = a x^\alpha \quad (1.7)$$

$$E = a x_0^\alpha \quad (1.8)$$

one obtains

$$T = \oint \sqrt{\frac{m}{2a x_0^\alpha (1 - x/x_0)^\alpha}} dx = \sqrt{\frac{m}{2a}} x_0^{1 - \frac{\alpha}{2}} \oint \frac{d\xi}{\sqrt{1 - \xi^\alpha}} \quad \xi = (x/x_0) \quad (1.9)$$

$$T \propto x_0^{1 - \frac{\alpha}{2}}$$

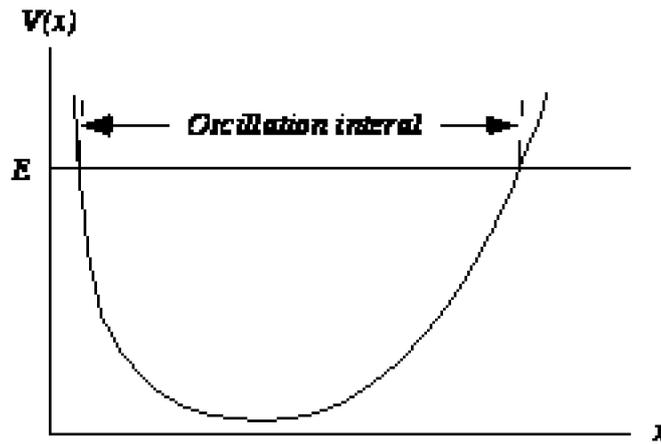


Fig. 1.1

For the harmonic oscillator, $\alpha = 2$. As a result, T does not depend on the amplitude. If a perturbation with a different power of x is added, say,

$$V(x) = x^2 + \gamma x^4 \tag{1.10}$$

then

- (i) T becomes dependent on amplitude;
- (ii) A mixing of scales takes place.

For small amplitude oscillations the potential is dominated by the x^2 term, motion is harmonic and T becomes almost independent of the amplitude. For large x , the x^4 term dominates making T amplitude dependent.

The issue of the determination of the period of oscillation is a central one. We know how to solve for harmonic problems, but in

$$\ddot{x} + \omega_0^2 x + f(x, \dot{x}) = 0 \tag{1.11}$$

an expansion around the unperturbed frequency, $T = (2\pi/\omega_0)$, will miss true periodicity. Striking examples can be found already in the linear world, such as the stroboscope effect, (e.g., the way in which car tires seem to turn at variable speed, forward or backward in films, depending on the mismatch between the true speed of rotation and the rate at which the frames are advanced by the movie projector).

1.2.2 Frequency multiplication-higher harmonics

Nonlinear problems have a tendency to multiply frequencies. If we are given that a solution of a particular problem has some frequency ω , then we expect higher multiples of ω to show up due to the nonlinearity. Consider, for instance,

$$\ddot{x} + \omega_0^2 x = \alpha x^3 \tag{1.12}$$

The nonlinearity will not only modify the frequency of the basic harmonic from ω_0 to some ω . Terms with frequencies that are integer multiples of ω will appear due to the nonlinear perturbation.

At least for small nonlinearities, we shall see that frequency multiplication is just a "cosmetic" effect, and that the core of the dynamics of the problem lies in the effect of the nonlinearity on the basic frequency of the solution and on the amplitude.

1.2.3 Finite speed of propagation in diffusion

In linear diffusion, say, heat conduction, the speed at which information propagates in space is infinite. That is, a signal, or a perturbation are spread out immediately over the whole space. For instance, consider the problem of heat diffusion over an infinite line with given initial conditions

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1.13)$$

The solution of this equation is

$$T(t, x) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{+\infty} e^{-\frac{(\xi-x)^2}{4\alpha t}} T_0(\xi) dx \quad (1.14)$$

Here, the effect of any point x at $t=0$ is immediately felt along the whole line, corresponding to an infinite speed of propagation. One important feature of nonlinear diffusion (which, as is pointed out in Section 1, covers a wide range of phenomena) is the finite speed at which signals propagate. Even in the relatively simple one-dimensional fluid equation (The famous Burgers equation)

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \alpha \frac{\partial^2 v}{\partial x^2}, \quad (1.15)$$

signals propagate at a finite speed.

1.3 Strong nonlinearities

Problems in which the nonlinearity is not just a small perturbation have become an area of extremely active research. Their solutions display new features that are completely different from what we are used to. Our intuition, based on the mathematical niceties of linear problems turns out to be almost useless.

1.3.1 Sensitivity to initial conditions or control parameters -

a headache to computers

Probably, the most important new feature is the tremendous sensitivity of the solutions of many nonlinear problems to slight variations in the initial conditions or in control parameters, to the extent that the approximate nature of numerical solutions on computers (owing to both round-off errors and to errors resulting from the discretized form of solution of continuous problems) renders them meaningless. The error at each time step of the calculation is only a small change in the solution. But the sensitivity to small changes causes the divergence of the numerical solution from the present into the future so that there may be no resemblance whatsoever between it and the actual evolution of the system considered. This is the origin of the "Butterfly effect": a butterfly flutters its wings in Beijing and a hurricane develops in New York. Intuitively, we do not expect this to happen, but on the computer - this may very well take place.... Similarly, a roundoff error in the typing in of a control parameter, may change the shape of the solution drastically, since many nonlinear systems exhibit "critical" behavior. Their solutions change shape completely when some control parameter crosses certain values.

Of no lesser importance is the fact that the sensitivity of the solution of many nonlinear problems to initial conditions makes the reproducibility of experiments highly unlikely, and the intervention of measuring instruments extremely undesired. Just imagine that an experiment is carefully designed in order to study turbulent flow in a fluid (a typical example of a multi-dimensional nonlinear system) contained in a vessel. The sensitivity to slight changes in initial (or boundary) conditions means that any uncontrollable fluctuation due, say, to surface roughness, or a minute change in temperature beyond the achievable accuracy, will make two consecutive experiments evolve quite differently. The distance between the points describing the two solutions in phase space will increase exponentially in time with an exponent (Liapounov exponent) characteristic of the particular problem. Consequently, in no time, the two trajectories of the same system will not resemble one another at all. A more extreme situation may evolve, in a fluid dynamics experiment where the behavior of the fluid depends on the Rayleigh number, for instance. Slight changes in the latter which are beyond the experimental accuracy may transform the system from a purely conductive mode to a convective mode with nicely shaped Rayleigh - Bénard cells, or from laminar convection into turbulent convection.

1.3.2. Chaos

Chaos, a feature of nonlinear systems, is intimately connected with their sensitivity to initial conditions as well as to changes in control parameters. Usually, when we observe the evolution in time of an "ordinary" dynamical system, we expect some regularity in its time dependence. That is, we expect the motion to be describable in terms of some simple functions, such as a Fourier expansion in trigonometric functions. When relevant parameters in a nonlinear problem take on some values, a new phenomenon takes place: observing the evolution of the solution over any period of time you wish, you will not discern any regularity in the behavior. The time dependence looks erratic despite the fact that it is a product of a set of completely deterministic differential equations. The motion looks so erratic, that, were it not for the fact that we know that the system follows well defined equations of motion, we might even be tempted to guess that this is a stochastic behavior. In continuous systems Chaos occurs only if they are at least three-dimensional. It occurs in lower dimensional discrete systems and in differential delay equations.

1.3.3 Strange attractors and how computers are saved

Luckily, in many cases, despite the tremendous sensitivity to slight changes in initial conditions or control parameters, the evolution of nonlinear systems does exhibit some regularities. Following the motion of the system in (the multi-dimensional) phase space, one sometimes finds that the "erratic" evolution is attracted asymptotically in time onto a subspace of a lower dimension. For example, if phase space is three-dimensional, then that subspace could be a two dimensional surface. On that subspace the "world line" describing the evolution of our dynamical system still exhibits the same features of high sensitivity to changes in initial conditions or control parameters and chaotic behavior. In addition, the collection of all the points through which the world line of the system passes is a strange creature. While they may be everywhere dense in that subspace (or a portion thereof) they still are of a lower "dimension" than that of the subspace. In fact, in some geometrical terms they are considered to have fractional dimension. (For instance, for a three-dimensional system attracted onto a two-dimensional subspace, the collection of all the points on that surface, through which the system passes may have a dimension less than 2 and more than 1 - it is "more" than a line, and "less" than a surface). Such an attracting set of points is called a strange attractor.

This is the saving grace for computers. While they are unable to predict numerically the precise trajectories of many nonlinear systems in phase space, the "incorrect" solutions they produce are attracted to the same strange attractors as the real systems do. Thus, a graphic representation of the evolution of a given nonlinear system in phase space based on a numerical solution may reveal to us the existence of strange attractors.

1.3.4 Chaos elsewhere - Feigenbaum and more

Chaos is not confined to continuous multi-dimensional dynamical systems only. In fact, it is much easier to discover in simple one-dimensional difference equations of the type

$$x_{n+1} = f(x_n, \mu) \quad (1.16)$$

provided $f(x, \mu)$ has certain properties. Here μ is a control parameter. The pioneering work of Feigenbaum on such equations [2] was the first place where phenomena such as period doubling and chaos were discovered. Only later on were they found numerically or experimentally in continuous systems.

The concept of period doubling will be dealt with later on. Here we just mention briefly that, in a certain range of the control parameter, the solution of the difference equation (1.16) exhibits a periodic behavior. That is, a set of two points x_0 and x_1 are images of each other under the transformation $f(x, \mu)$. When μ is increased beyond a certain critical value, the set of two points suddenly splits into a set of four points which are images of one another in a periodic sequence: $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_0 \rightarrow \dots$. The period is doubled (2 goes into 4). Further on, one finds another critical value of μ beyond which the period doubles again, this time the period is 8. A sequence of values of μ exists, such that, when μ crosses each of them, the period of the solution doubles. This sequence converges onto a critical value, μ_∞ , beyond which the solution does not exhibit any periodicity. It is chaotic.

The phenomenon of period doubling has since then been discovered in continuous systems numerically (e.g. the Lorenz equations [3]) and experimentally in the study of the onset of convection and turbulence in fluids [4] which has become possible with the advent of lasers.

Another type of phenomena where features of the type discussed here appear, are ones governed by delay differential equations, even in one dimension. The solutions of equations of the types

$$\frac{dx}{dt} = f(x(t - \tau)) \quad (1.17a)$$

$$\varepsilon \frac{dx}{dt} = f(x(t + \tau)) - g(x(t)) \quad (1.17b)$$

exhibit features similar to those exhibited by "ordinary" nonlinear systems [5].

1.4 Why perturbation methods are important

A major part of this course will be devoted to the development of perturbative treatments for nonlinear problems. The reasons for the stress of perturbation methods are as follows. First, in many nonlinear problems the nonlinearity is small. One example is the production of sound waves (not shock waves!) by vibrating objects. Another is the description of the motion of atoms in a crystal, where they are typically located at the minimum of the potential well of their mutual attraction, and the force is very close to harmonic. In addition, strong nonlinearities are rarely amenable to an analytical treatment, and most of what we know about chaos and strange attractors is based on numerical "experiments". Last but not least is the fact that the analysis of critical behavior (the onset of instabilities in nonlinear systems) can be quite often carried out in approximation schemes in which the nonlinearity of the resulting equations is small, even when the original equations for the problem studied may be strongly nonlinear. An example for this case is the study of the onset of convection in fluids. The equations of fluid dynamics can be looked upon as strongly nonlinear. However, when the Rayleigh number, Ra (which is a dimensionless measure of the heating rate), is just slightly above the critical value, Ra_c , separating pure conduction from convection, the onset of convection is characterized by small velocities related to the small parameter $(Ra - Ra_c)$. The equations for the small amplitude of the velocity field can be expanded in this small parameter so that the resulting nonlinearity turns out to be a small perturbation. Thus, while the full

solution of the fluid motion for all values of Ra requires numerical techniques, the analysis of the onset of the new phenomenon can be carried out in a perturbation scheme.

1.5 Departures from linear theory

Nonlinear equations and their solutions exhibit departures from the common features of the linear theory that we are used to. To start with, if we formally write a differential equation as

$$Lg = 0$$

where L is the operator and g is the function operated on, then in a linear problem we have

$$L(\alpha g_1 + \beta g_2) = \alpha Lg_1 + \beta Lg_2 \quad (1.18)$$

where α and β are constants. For instance, if $L \equiv d^2/dx^2 + \cos x$, then

$$L(\alpha g_1 + \beta g_2) = \alpha \left(\frac{d^2 g_1}{dx^2} + \cos x g_1 \right) + \beta \left(\frac{d^2 g_2}{dx^2} + \cos x g_2 \right)$$

If L is a nonlinear operator, then Eq. (1.18) does not hold. For instance, in the nonlinear equation

$$\frac{d^2 g}{dx^2} + g + \epsilon g^3 = 0$$

we have

$$\begin{aligned} L(\alpha g_1 + \beta g_2) &= \frac{d^2(\alpha g_1 + \beta g_2)}{dx^2} + (\alpha g_1 + \beta g_2) + \epsilon(\alpha g_1 + \beta g_2)^3 \\ &= \alpha Lg_1 + \beta Lg_2 + 3\epsilon(\alpha^2 \beta g_1^2 g_2 + \alpha \beta^2 g_1 g_2^2) \end{aligned}$$

Thus, *the principle of superposition does not hold for nonlinear problems.* An immediate conclusion is that the notion of *linearly independent solutions* loses its meaning. For example, the second order linear problem

$$\ddot{y} + y = 0$$

has two linearly independent solutions, namely, $\sin t$ and $\cos t$. The most general solution of the problem is a linear combination of these two solutions. This is not the case in nonlinear problems.

The third deviation is the dependence on *integration constants and/or initial conditions*. The solutions of linear equations are linear functions of the latter. This is not the case in nonlinear problems. As an example, consider the equation

$$x \ddot{x} + \dot{x}^2 = 0 \quad (1.19)$$

which is solved by

$$x = \pm \sqrt{At + B}$$

The dependence on the two integration constants A and B is nonlinear, and specific to the equation. In particular, if x is a solution of a nonlinear problem, then, in general, $a \cdot x$ is not a solution, unless $a=1$, as is easily seen in the example of Eq. (1.19).

Exercises

1.1 For the delay differential equation

$$\frac{dx}{dt} = -\lambda x(t - \tau)$$

(τ is a constant time delay) find the poles of the Laplace transform of the solution $x(t)$. Discuss the feature of the solution that makes it so different from the solution of the same equation without time delay.

1.2 A differential equation,

$$a_n(x) \left(\frac{d^n x}{dt^n} \right)^{k_n} + a_{n-1}(x) \left(\frac{d^{n-1} x}{dt^{n-1}} \right)^{k_{n-1}} + \dots + a_0(x) = 0$$

is called *homogeneous of degree m* if all the terms in the equation are monomials of degree m. Show that despite the fact that the equation is nonlinear, if $x(t)$ is a solution, so is $\alpha \cdot x(t)$, with α constant.