THE MATHIEU EQUATION

The equation:

$$ \tag{1} $$

The expansion:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \qquad a = \tfrac{1}{4} + \varepsilon a_1 + \varepsilon^2 a_2 + \varepsilon^3 a_3 + \cdots \tag{2}$$

Notation:

$$D_n \equiv \frac{\partial}{T_n} \qquad T_n \equiv \varepsilon^n t \qquad n = 0,1,2,3,\ldots \tag{3}$$

The solvability conditions:

<u>O(ε)</u>

$$\left.\begin{aligned} D_1 A &= i a_1 A + i \overline{A} \\ D_1 \overline{A} &= -i A - i a_1 \overline{A} \end{aligned}\right\} \quad \Rightarrow \quad D_1 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = \left(i a_1 \sigma_z - \sigma_y \right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{4}$$

The σ's are the Pauli matrices. Eq. (4) leads to

$$D_1^{\,2} \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = -\left(a_1^{\,2} - 1 \right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{5}$$

Stability requires

$$|a_1| > 1 \tag{6}$$

The higher-order solvability conditions can be also written in terms of the σ's.

<u>O(ε²)</u>

$$\begin{aligned} D_1 \begin{pmatrix} B \\ \overline{B} \end{pmatrix} &= \left(i a_1 \sigma_z - \sigma_y \right) \begin{pmatrix} B \\ \overline{B} \end{pmatrix} - D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + i \sigma_z D_1^{\,2} \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + i \left(a_1 + \tfrac{1}{2} \right) \sigma_z \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \\ &= \left(i a_1 \sigma_z - \sigma_y \right) \begin{pmatrix} B \\ \overline{B} \end{pmatrix} - D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + i g \sigma_z \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \\ &\left(g = \left[\left(1 - a_1^{\,2} \right) + a_2 + \tfrac{1}{2} \right] \right) \end{aligned} \tag{7}$$

[Using Eq. (5) yields the final form of the r.h.s. of Eq. (7).] Eqs. (4), (5) & (7) lead to

$$D_1^{\,2}\begin{pmatrix} B \\ \overline{B} \end{pmatrix} + \left(a_1^{\,2} - 1\right)\begin{pmatrix} B \\ \overline{B} \end{pmatrix} = -2\left(i\,a_1\,\sigma_z - \sigma_y\right) D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} - 2\,a_1\,g \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{8}$$

***O*(ε³)**

$$\begin{aligned} D_1\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = &\left(i\,a_1\,\sigma_z - \sigma_y\right)\begin{pmatrix} C \\ \overline{C} \end{pmatrix} - D_2\begin{pmatrix} B \\ \overline{B} \end{pmatrix} + i\,\sigma_z\, D_1^{\,2}\begin{pmatrix} B \\ \overline{B} \end{pmatrix} + i\left(a_1 + \tfrac{1}{2}\right)\sigma_z \begin{pmatrix} B \\ \overline{B} \end{pmatrix} \\ &- D_3\begin{pmatrix} A \\ \overline{A} \end{pmatrix} - 2\left(a_1 + \sigma_x\right) D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + \left(i\left(a_3 - \tfrac{1}{2}\right)\sigma_z + \tfrac{3}{4}\sigma_y\right)\begin{pmatrix} A \\ \overline{A} \end{pmatrix} \end{aligned} \tag{9}$$

Eqs. (4), (5), (7), (8) & (9) lead to

$$\begin{aligned} D_1^{\,2}\begin{pmatrix} C \\ \overline{C} \end{pmatrix} + \left(a_1^{\,2} - 1\right)\begin{pmatrix} C \\ \overline{C} \end{pmatrix} = &-2\left(i\,a_1\,\sigma_z - \sigma_y\right) D_2 \begin{pmatrix} B \\ \overline{B} \end{pmatrix} - 2\,a_1\,g \begin{pmatrix} B \\ \overline{B} \end{pmatrix} \\ &-2\left(i\,a_1\,\sigma_z - \sigma_y\right) D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \\ &+D_2^{\,2}\begin{pmatrix} A \\ \overline{A} \end{pmatrix} - 2\,i\,g\,\sigma_z\, D_2\begin{pmatrix} A \\ \overline{A} \end{pmatrix} - \left\{g^2 + 2\,a_1\left(a_3 - 2\,a_1\,g - \tfrac{1}{2}\right) + \tfrac{3}{2}\right\}\begin{pmatrix} A \\ \overline{A} \end{pmatrix} \end{aligned} \tag{10}$$

Particular choice

The free amplitudes, *B*, *C* obey the homogeneous parts of Eqs. (7) & (9), respectively. (This includes, in particular, the choice of *no free amplitudes*: $B = C = 0$.) Eqs. (7) & (9) then yield:

$$D_2\begin{pmatrix} A \\ \overline{A} \end{pmatrix} = i\,g\,\sigma_z \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{11}$$

$$D_3\begin{pmatrix} A \\ \overline{A} \end{pmatrix} = -2\left(a_1 + \sigma_x\right) D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + \left(i\left(a_3 - \tfrac{1}{2}\right)\sigma_z + \tfrac{3}{4}\sigma_y\right)\begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{12}$$

Consistency of Eqs. (4) and (11): Apply D_2 to Eq. (4) and D_1 - to Eq. (11). For the results to coincide

$$g = 0 \quad \Rightarrow \quad \left(1 - a_1^{\,2}\right) + \left(a_2 + \tfrac{1}{2}\right) = 0 \tag{13}$$

must be obeyed. At the boundary of the stability domain, Eq. (13) leads to

$$a_2 \xrightarrow[a_1^2 \to 1]{} -\tfrac{1}{2} \tag{14}$$

With $g = 0$, Eq. (11) becomes

$$D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = 0 \tag{15}$$

Hence, Eq. (12) is reduced to

$$D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = \left(i\left(a_3 - \tfrac{1}{2}\right)\sigma_z + \tfrac{3}{4}\sigma_y \right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{16}$$

Consistency of Eqs. (4) and (16): Apply D_3 to Eq. (4) and D_1 - to Eq. (16). For the results to coincide

$$a_1\left(a_3 - \tfrac{1}{2}\right) + \tfrac{3}{4} = 0 \tag{17}$$

must be obeyed. Hence, at the boundary of the stability domain, Eq. (17) leads to

$$a_3 \to \begin{cases} -\tfrac{1}{4} & a_1 \to +1 \\ +\tfrac{5}{4} & a_1 \to -1 \end{cases} \tag{18}$$

General case

Consider Eq. (8). To avoid T_1 - secular terms in B, so that B is periodic in T_1, the r.h.s. of Eq. (8) must vanish:

$$\begin{aligned} &\left(i a_1 \sigma_z - \sigma_y\right) D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + a_1 g \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = 0 \quad \Rightarrow \quad D_2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = \frac{a_1 g}{a_1^2 - 1}\left(i a_1 \sigma_z - \sigma_y\right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \\ &\Rightarrow \quad D_2^2 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = -\frac{\left(a_1 g\right)^2}{a_1^2 - 1} \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \end{aligned} \tag{19}$$

Eqs. (4) & (19) are solved by

$$\begin{pmatrix} A \\ \overline{A} \end{pmatrix} = \exp\left[\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \mathbf{A}_0\left(T_2, T_3, \ldots\right) = \exp\left[\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \exp\left[\frac{a_1 g}{a_1^2 - 1}\left(i a_1 \sigma_z - \sigma_y\right) T_2\right] \mathbf{A}_{00}\left(T_3, \ldots\right)$$

$$= \left\{\cos\left(\sqrt{a_1^2 - 1}\, T_1\right) + \frac{\left(i a_1 \sigma_z - \sigma_y\right)}{\sqrt{a_1^2 - 1}} \sin\left(\sqrt{a_1^2 - 1}\, T_1\right)\right\} \times$$

$$\left\{\cos\left(\frac{a_1 g}{\sqrt{a_1^2 - 1}} T_2\right) + \frac{\left(i a_1 \sigma_z - \sigma_y\right)}{\sqrt{a_1^2 - 1}} \sin\left(\frac{a_1 g}{\sqrt{a_1^2 - 1}} T_2\right)\right\} \mathbf{A}_{00}\left(T_3, \ldots\right)$$

(20)

To guarantee that A is bounded as $a_1{}^2$ tends to 1, one needs

$$g = \underset{\left(a_1^2 \to 1\right)}{O\left(a_1^2 - 1\right)} \qquad a_2 \xrightarrow[a_1^2 \to 1]{} -\tfrac{1}{2} + O\left(a_1^2 - 1\right) \tag{21}$$

Eqs. (13) & (14) constitute a particular case of the more general requirement of Eq. (21).

Write

$$\begin{pmatrix} B \\ \overline{B} \end{pmatrix} = \exp\left[\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \tilde{\mathbf{B}}(T_1)$$
$$= \left\{\cos\left(\sqrt{a_1^2 - 1}\, T_1\right) + \frac{\left(i a_1 \sigma_z - \sigma_y\right)}{\sqrt{a_1^2 - 1}} \sin\left(\sqrt{a_1^2 - 1}\, T_1\right)\right\} \tilde{\mathbf{B}}(T_1) \tag{22}$$

Eqs. (7) & (19) yield

$$D_1 \tilde{\mathbf{B}}(T_1) = \frac{g}{a_1^2 - 1} \exp\left[-\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \sigma_x \left(i a_1 \sigma_z - \sigma_y\right) \exp\left[\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \mathbf{A}_0\left(T_2\right)$$
$$= \frac{g}{a_1^2 - 1} \sigma_x \left(i a_1 \sigma_z - \sigma_y\right) \exp\left[2\left(i a_1 \sigma_z - \sigma_y\right) T_1\right] \mathbf{A}_0\left(T_2\right) \tag{23}$$

$$\tilde{\mathbf{B}}(T_1) = \tfrac{1}{2}\frac{g}{a_1^{\,2}-1}\sigma_x \exp\left[2\left(ia_1\sigma_z-\sigma_y\right)T_1\right]\mathbf{A}_0(T_2)+\mathbf{B}_0(T_2) \tag{24}$$

Using Eq. (24) in Eq. (22), one obtains

$$\begin{aligned}\begin{pmatrix} B \\ \overline{B}\end{pmatrix} = \tfrac{1}{2}\frac{g}{a_1^{\,2}-1}\sigma_x &\left\{\cos\left(\sqrt{a_1^{\,2}-1}\,T_1\right)+\frac{\left(ia_1\sigma_z-\sigma_y\right)}{\sqrt{a_1^{\,2}-1}}\sin\left(\sqrt{a_1^{\,2}-1}\,T_1\right)\right\}\mathbf{A}_0(T_2) \\ &+\left\{\cos\left(\sqrt{a_1^{\,2}-1}\,T_1\right)+\frac{\left(ia_1\sigma_z-\sigma_y\right)}{\sqrt{a_1^{\,2}-1}}\sin\left(\sqrt{a_1^{\,2}-1}\,T_1\right)\right\}\mathbf{B}_0(T_2)\end{aligned} \tag{25}$$

Eq. (21) ensures that B does not blow up when $a_1{}^2$ tends to 1.

Now consider Eq. (10). Employing Eq. (19), Eq. (10) becomes

$$\begin{aligned}&D_1^{\,2}\begin{pmatrix} C \\ \overline{C}\end{pmatrix}+\left(a_1^{\,2}-1\right)\begin{pmatrix} C \\ \overline{C}\end{pmatrix}= \\ &-2\left(ia_1\sigma_z-\sigma_y\right)D_2\begin{pmatrix} B \\ \overline{B}\end{pmatrix}-2a_1 g\begin{pmatrix} B \\ \overline{B}\end{pmatrix} \\ &-2\left(ia_1\sigma_z-\sigma_y\right)D_3\begin{pmatrix} A \\ \overline{A}\end{pmatrix}-\left\{2a_1\left(a_3-2a_1 g-\tfrac{1}{2}\right)+\tfrac{3}{2}\right\}\begin{pmatrix} A \\ \overline{A}\end{pmatrix}+2\frac{g^2}{a_1^{\,2}-1}\left(a_1\sigma_x+\tfrac{1}{2}\right)\begin{pmatrix} A \\ \overline{A}\end{pmatrix}\end{aligned} \tag{26}$$

To avoid T_1 - secular terms in C, so that C is periodic in T_1, the r.h.s. of Eq. (26) must vanish:

$$\begin{aligned}&-\left(ia_1\sigma_z-\sigma_y\right)D_2\begin{pmatrix} B \\ \overline{B}\end{pmatrix}-a_1 g\begin{pmatrix} B \\ \overline{B}\end{pmatrix} \\ &-\left(ia_1\sigma_z-\sigma_y\right)D_3\begin{pmatrix} A \\ \overline{A}\end{pmatrix}-\left\{a_1\left(a_3-2a_1 g-\tfrac{1}{2}\right)+\tfrac{3}{4}\right\}\begin{pmatrix} A \\ \overline{A}\end{pmatrix}+\frac{g^2}{a_1^{\,2}-1}\left(a_1\sigma_x+\tfrac{1}{2}\right)\begin{pmatrix} A \\ \overline{A}\end{pmatrix}=0\end{aligned} \tag{27}$$

Eq. (27) leads to

$$\begin{aligned}D_2\begin{pmatrix} B \\ \overline{B}\end{pmatrix}=\frac{a_1 g}{a_1^{\,2}-1}\left(ia_1\sigma_z-\sigma_y\right)\begin{pmatrix} B \\ \overline{B}\end{pmatrix}-D_3\begin{pmatrix} A \\ \overline{A}\end{pmatrix}-\frac{a_1 g^2}{\left(a_1^{\,2}-1\right)^2}\left(ia_1\sigma_z-\sigma_y\right)\sigma_x\begin{pmatrix} A \\ \overline{A}\end{pmatrix}+h\left(ia_1\sigma_z-\sigma_y\right)\begin{pmatrix} A \\ \overline{A}\end{pmatrix} \\ \left(h=\tfrac{1}{2}\frac{g^2}{\left(a_1^{\,2}-1\right)^2}-\frac{a_1\left(a_3-2a_1 g-\tfrac{1}{2}\right)+\tfrac{3}{4}}{a_1^{\,2}-1}\right)\end{aligned} \tag{28}$$

Employing Eq. (19), Eq. (28) yields

$$D_2^{\,2}\begin{pmatrix} B \\ \overline{B} \end{pmatrix} + \frac{a_1^{\,2} g^2}{a_1^{\,2} - 1}\begin{pmatrix} B \\ \overline{B} \end{pmatrix} = -2\frac{a_1 g}{a_1^{\,2} - 1}\left(i\, a_1 \sigma_z - \sigma_y\right) D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} - 2 h\, a_1 g \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{29}$$

To avoid T_2 - secular terms in B, so that B is periodic in T_2, the r.h.s. of Eq. (29) must vanish:

$$-\frac{a_1 g}{a_1^{\,2} - 1}\left(i\, a_1 \sigma_z - \sigma_y\right) D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} - h a_1 g \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = 0 \tag{30}$$

Eq. (30) is solved by $g = 0$, which yields Eqs. (13) & (14) of the **particular case**, discussed on p. 3, or by

$$D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} = h\left(i\, a_1 \sigma_z - \sigma_y\right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{31}$$

Consider, first, Eq. (31), which implies

$$\begin{pmatrix} A \\ \overline{A} \end{pmatrix} \propto \exp\left\{h\left(i\, a_1 \sigma_z - \sigma_y\right) T_3\right\} = \cos\left(\frac{h}{\sqrt{a_1^{\,2} - 1}} T_3\right) + \frac{\left(i\, a_1 \sigma_z - \sigma_y\right)}{\sqrt{a_1^{\,2} - 1}} \sin\left(\frac{h}{\sqrt{a_1^{\,2} - 1}} T_3\right) \tag{32}$$

To guarantee that A is bounded as a_1^2 tends to 1, one needs

$$h = O\left(a_1^{\,2} - 1\right) \tag{33}$$

The definition of h [see Eq. (28)], implies that as a_1^2 tends to 1, one must have

$$a_1\left(a_3 - \tfrac{1}{2}\right) + \tfrac{3}{4} = O\left(a_1^{\,2} - 1\right) \tag{34}$$

Eq. (34) yields Eq. (18) for the value of a_3 at the boundary of the stability domain.

If the alternative of $g = 0$ is chosen for satisfying Eq. (30), then Eq. (19) is reduced to Eq. (15) and Eq. (28) becomes

$$D_2 \begin{pmatrix} B \\ \overline{B} \end{pmatrix} = -D_3 \begin{pmatrix} A \\ \overline{A} \end{pmatrix} + h\left(i\, a_1 \sigma_z - \sigma_y\right) \begin{pmatrix} A \\ \overline{A} \end{pmatrix} \tag{35}$$

Then, owing to Eq. (15), B will have a T_2 - secular term, unless Eq. (31) is obeyed, leading, again, to Eqs. (33) & (34).

Comment

The results for the coefficients a_2, a_3, ..., *inside* the stability domain depend on the choice of the free amplitudes, B, C, ..., because the latter depend on g, h, ... However, if expressed consistently (in powers of ε) in terms of the *initial conditions*, they would not depend on the choice of these amplitudes. The boundary of the stability domain is a property of the original equation, hence cannot depend on the choice of the free amplitudes.

Comment

In the Normal Form analysis of the Mathieu equation (e.g., in the book by Peter Kahn & myself) we wrote for a of Eq. (1)

$$a = \tfrac{1}{4} + \mu \qquad |\mu| \ll 1 \tag{35}$$

One performs an expansion in the two *independent* small parameters ε and μ. The second-order analysis yields for the stability domain

$$-\varepsilon - \tfrac{1}{2}\varepsilon^2 \le \mu \le \varepsilon - \tfrac{1}{2}\varepsilon^2 \tag{36}$$

There are no limitations on a_1 or a_2, ..., except at the boundary of the stability domain, where Eq. (36) yields $a_1 = \pm 1$ and $a_2 = -1/2$. The intuitive significance of this result is that, within the stability domain in the $\mu - \varepsilon$ plane, one may correlate μ and ε in any desired manner. Note that this result does not depend on whether one includes free amplitudes or chooses to set them to zero.

In the MMTS analysis that treats μ and ε as independent parameters, the requirement of commuting partial derivatives with respect to different time scales when applied to an amplitude Q:

$$D_n D_m Q = D_m D_n Q \tag{37}$$

leads to the conclusion that the only solution that solves Eq. (1) *consistently* is $x(t)=0$. This is why μ and ε have to be correlated.

Correlating a and μ [Eq. (2)] as in the analysis carried out here, if the free amplitudes are set to zero, then the consistency requirements relate a_2, a_3, ... to a_1. Namely, the consistent analysis has to be carried out along lines within the $\mu - \varepsilon$ plane, each specified by a choice of a_1. The whole stability domain is covered by varying a_1 over its allowed range.

If the free amplitudes are included, the additional freedom leads to a relaxation of the constraints on the coefficients a_2, a_3, ... As in the normal form analysis, they are not constrained any more, except at the boundary of the stability domain.

<u>Comment</u>

Bender & Orszag did notice that a problem arises as $a_1{}^2$ tends to 1, and arrived at the conclusion that the actual small parameter ought to be $\varepsilon^{1/2}$. I repeated the analysis detailed above for such an expansion through $O(\varepsilon^2)$:

$$\begin{aligned} &x = x_0 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \varepsilon^{3/2} x_{3/2} + \varepsilon^2 x_2 \\ &a = \tfrac{1}{4} + \varepsilon^{1/2} a_{1/2} + \varepsilon a_1 + \varepsilon^{3/2} a_{3/2} + \varepsilon^2 a_2 \\ &D_n \equiv \frac{\partial}{T_n} \qquad T_n \equiv \varepsilon^n t \qquad n = 0, \tfrac{1}{2}, 1, \tfrac{1}{2}, 2 \end{aligned} \tag{35}$$

for the **particular case** in which the free amplitudes obey the homogeneous parts of the solvability conditions. [See p. 3.] The results for a_1 and a_2 *coincide* with Eqs. (6), (13) & (14) and one finds $a_{1/2} = a_{3/2} = 0$. In addition, the amplitudes do not depend on $T_{1/2}$ and $T_{3/2}$. Moreover, $x_{1/2}$ and $x_{1/2}$ turn out to be trivial. They contain only *free* amplitudes. That is, they are solutions of the homogenous equation in T_0. Thus, at least with the special choice of the free functions, there is no need to introduce powers of $\varepsilon^{1/2}$.