On the road to chaos

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Submitted June 2003 *accepted* January 2004

Abstract

Two simple mathematical models for how individual vehicles follow each other along a stretch of road are discussed. The resulting difference equations can be used as applications of techniques taught at A-level and first year undergraduate level, and as an introduction to the behaviour of the logistic map.

1. Introduction

Traffic is of interest to student and schoolteacher alike. Both are aware of the many practical problems, such as a traffic jam making them late for school, the use of congestion charges to stop traffic jams happening in the first place or the irritating traffic light sequencing at the end of their road, which appears to keep them waiting for an eternity while other drivers move smoothly across the junction wearing what for all the world look like smug grins.

Academics are also interested in traffic, but in line with their boffin image they express their views by using equations and graphs and computer simulations to try and model mathematically how traffic behaves: Why do jams occur? What level of congestion charging will reduce congestion in the city centre effectively without simply moving the whole problem to the suburbs? How should the light sequencing at junctions be adjusted to keep the flow across it moving smoothly and hence leave all the drivers feeling moderately happy?

The models that are used to study traffic flow range from treating traffic as a compressible fluid [1] or as a kinetic gas [2] (called macroscopic models) to detailed (so called microscopic) models of how individual vehicles and their drivers behave [3].

In this paper we consider the numerical solution of a simple microscopic car following model where the aim of the driver is to follow the vehicle in front [4]. The numerical solution of this model can be interpreted as a new mathematical model for the car following process [5]. This new model can be used to simulate the behaviour of a driver who has lost control of his vehicle and uses mathematical techniques such as the Euler method, series summation and proof by induction, and also serves as an introduction to the behaviour of the logistic map. Thus, it is hoped that, like the real thing, this model may be of interest to both teachers and students as a motivator for the study of the logistic map, an application of techniques already taught, or as an example of simple model construction and interpretation.

2. Simple car following models

2.1. The Quick-Thinking Driver

In a typical formulation for a car following model [3] the driver of the following car adjusts her speed according to the relative velocity between her car and the car in front. This can be described by

$$\frac{du(t+T)}{dt} = \lambda[U(t) - u(t)] \tag{1}$$

where $U(t)$ is the velocity of the lead car at time t and $u(t)$ is the velocity of the following car at time t. T is the reaction or thinking time of the following driver. This is the time required for the driver in the following car to respond to a change in the lead car's behaviour. The sensitivity coefficient λ is a measure of how strongly the following driver responds to the approach/recession of the vehicle in front. The larger its value, the larger the reaction of the following driver to the relative velocity between vehicles. Experiments carried out by the American company General Motors in the 1950s [3] found λ to lie in the region of 0.3–0.4 s^{-1}. In more general formulations of equation (1), λ is a function of the spacing between the two cars and the following car's velocity.

As it stands, the solution of equation (1) can be investigated by using Laplace transform methods. However, we can simplify equation (1) by assuming that the driver of the following car thinks extremely quickly by setting $T = 0$. This gives rise to the so called Quick Thinking Driver model [4];

$$\frac{du(t)}{dt} = \lambda[U(t) - u(t)] \tag{2}$$

Given the functional form of the velocity of the lead car, $U(t)$, we can solve equation (2) to find the behaviour, $u(t)$, of the following car.

2.2. The Inattentive Driver model

Equation (2), with suitable initial conditions can be easily solved for a wide range of lead car behaviours $U(t)$ using the integrating factor to give

$$u(t) = \lambda e^{-\lambda t} \int e^{\lambda t} U(t) dt \tag{3}$$

However, let us suppose for a moment that we want to solve equation (2) numerically.

The simplest numerical method to use is the Euler method, and for a step size Δt the Euler method representation of equation (2) is,

$$u_{j+1} = u_j + \Delta t \lambda (U_j - u_j) \tag{4}$$

where

$$\begin{aligned} u_j &= u(\Delta t j) \\ U_j &= U(\Delta t j) \end{aligned} \tag{5}$$

An obvious objection to the use of the Euler method is that it is relatively inaccurate; however, a closer look at (4)–(5) reveals it to be a model for car following in its own right. In this new model the driver updates her acceleration every Δt time units. Consider the jth update, which occurs at time Δtj: the driver observes the difference in velocity between her vehicle and the one in front to be $(U_j - u_j)$. Having made this observation she then adjusts her acceleration to the value $\lambda(U_j - u_j)$ and maintains this constant acceleration for a time period Δt, until her next observation, by which time her velocity is $u_{j+1} = u_j + \Delta t\lambda(U_j - u_j)$. Thus, Euler's method, (4)–(5), can be thought of as a repeated application of the well known equation of motion for constant acceleration

$$v = u + at \tag{6}$$

where u is the initial velocity, a is the acceleration and v is the velocity after time t.

We call (4)–(5) the Inattentive Driver model, and a more general form of this model has been investigated elsewhere [5]. In the Inattentive Driver model the driver updates her behaviour (or, to put it another way, only pays attention to what is happening on the road) every Δt time units. As we let $\Delta t \to 0$ we recover the Quick-Thinking Driver model (2).

Thus, by using the Euler method to solve our differential equation, we have created a new model for car following and arguably this new model contains more aspects of reality than the original model. In the Quick-Thinking Driver model the driver is *continuously* and *smoothly* updating his acceleration to keep pace with the driver in front. This is undoubtedly *not* what real drivers do, as it seems reasonable that real drivers update their behaviour at (albeit short) intervals and alter their accelerations in a non-smooth way.

3. Simple car following

Consider a stationary vehicle. At time $t = 0$ a car moving at constant velocity U drives past a stationary vehicle, which immediately begins to follow it. Using the Inattentive Driver model, what is the subsequent motion of the following vehicle?

This problem amounts to solving equations (4) and (5) with $U_j = U \ \forall j \geq 0$ and $u_0 = 0$. We can rewrite (4) and (5) as

$$u_{j+1} = \beta u_j + (1 - \beta)U \tag{7}$$

where

$$\beta = 1 - \lambda\Delta t. \tag{8}$$

Using $u_0 = 0$ and repeated application of (7) yields

$$\left.\begin{aligned} u_1 &= U(1-\beta) \\ u_2 &= U(1-\beta^2) \\ u_3 &= U(1-\beta^3) \end{aligned}\right\} \tag{9}$$

This suggests that the general solution of (7) is

$$u_n = U(1 - \beta^n) \tag{10}$$

a result which is easily proved by induction.

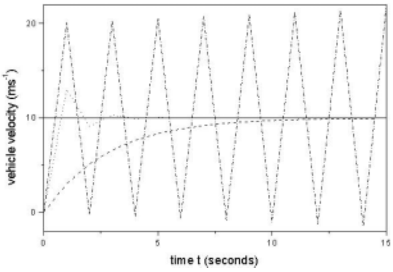

Fig 1. Convergent and divergent behaviour of equation (10) with lead vehicle velocity $U = 10\,\mathrm{ms}^{-1}$ (solid line). The other lines show the behaviour of the following car whose driver updates his behaviour every $\Delta t = 1\,\mathrm{s}$ for various choices of sensitivity coefficient λ. Dashed line: $\lambda = 0.3\,\mathrm{s}^{-1}$, $\beta = 1 - \lambda\Delta t = 0.7$ giving monotonic convergence. Dotted line: $\lambda = 1.3\,\mathrm{s}^{-1}$, $\beta = 1 - \lambda\Delta t = -0.3$ giving oscillatory convergence. Dash-dot line: $\lambda = 2.01\,\mathrm{s}^{-1}$, $\beta = 1 - \lambda\Delta t = -1.01$ giving divergent behaviour.

Clearly if $|\beta| > 1$, u_n diverges as $n \to \infty$, whereas if $0 < |\beta| < 1$ then

$$\lim_{n\to\infty} u_n = U. \tag{11}$$

Further, if $0 < \beta < 1$ then u_n converges to U monotonically, whereas if $-1 < \beta < 0$ then u_n converges to U in an oscillatory fashion. The behaviour of equation (10) is illustrated in Fig. 1 for various choices of λ. We would expect a safe driver to converge monotonically to the velocity of the lead driver, whereas a driver who is not safe to be on the roads will converge in oscillatory fashion. We ignore the divergent case as it leads to unphysical negative velocities where the driver in the following vehicle brakes so hard that they end up moving backwards! Thus, we can describe a driver as safe if $0 < \lambda\Delta t < 1$ and unsafe if $1 < \lambda\Delta t < 2$.

The fact that the controlling term is the *product* of λ and Δt is significant, since a driver who, for example, has taken alcohol is not only going to be less attentive (i.e. have a larger than average value of Δt) but will also tend to overreact to visual stimuli (i.e. have a larger than average value of λ) thus leading to 'doubly' increased likelihood of unsafe behaviour.

The question of whether such unsafe driving will lead to a collision is left as an exercise for the reader (see section 5).

3.1. Chaos on the roads

A simple generalisation of this model is to make the sensitivity coefficient λ of the following driver dependent on her velocity thus giving a new Quick-Thinking Driver model,

$$\frac{du(t)}{dt} = \gamma u(t)[U(t) - u(t)], \tag{12}$$

where γ is our new sensitivity coefficient. We can estimate the value of γ by assuming that under normal steady driving conditions where the following car is moving at 45 km/h (i.e. $\sim 13\,\mathrm{ms}^{-1}$ or 30 mph) the models as described by equations (2) and (12) should predict the same acceleration for the same relative velocity i.e. $13\gamma = \lambda$. Thus, given λ typically lies between 0.3 and 0.4 s^{-1}, we expect typical values of γ to be in the region of 0.02 to 0.03 s^{-1}.

The inattentive driver model corresponding to equation (12) is given by

$$u_{j+1} = u_j + \Delta t \gamma u_j (U_j - u_j) \tag{13}$$

where U_j and u_j are as defined by equation (5). If we consider again the case of a car following a lead vehicle which passes it at $t = 0$ with constant velocity U then equation (13) can be written in the form of the logistic map [6]

$$v_{j+1} = a v_j (1 - v_j) \tag{14}$$

where

$$a = 1 + \gamma U \Delta t \tag{15}$$

and

$$v_j = \frac{\gamma \Delta t}{1 + \gamma U \Delta t} u_j. \tag{16}$$

The most important feature of the logistic map is the control parameter a. Its value is crucial to the evolution of v_j. The sequences of solutions generated by the logistic map are called orbits. Examples of the solution of equation (14) for various values of a are given in Fig. 2 where the starting value, $v_0 = 0.3$ in each case. We note that if

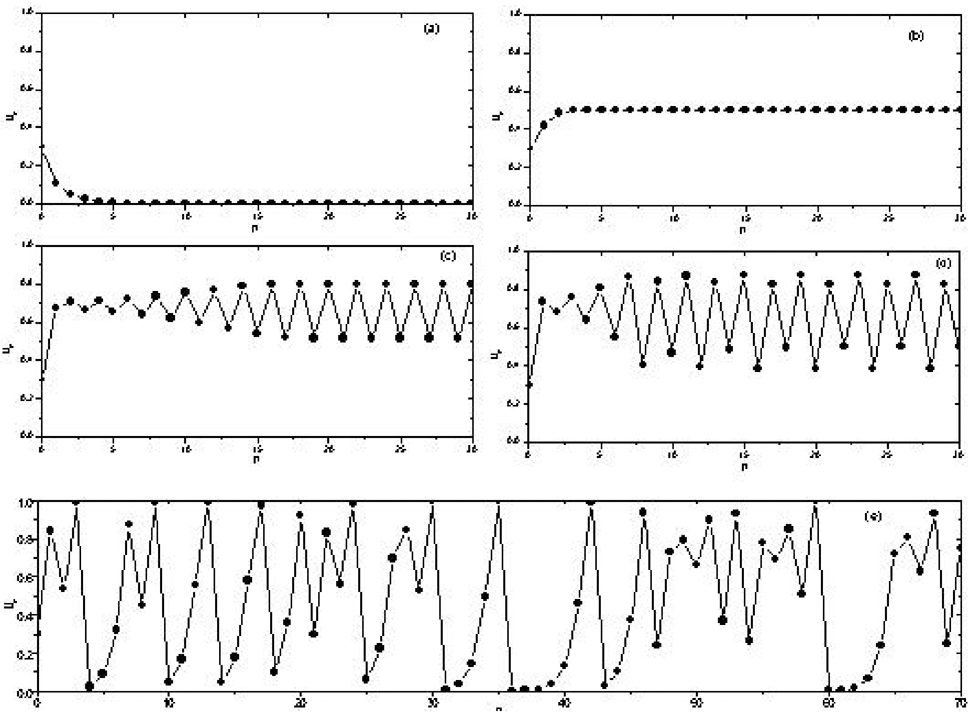

Fig 2. Solutions of the logistic equation (**14**) for various values of the control parameter a: (a) $a = 0.5$, (b) $a = 2$, (c) $a = 3.2$, (d) $a = 3.5$, (e) $a = 4$. In each case the starting value $v_0 = 0.3$.

$0 < a \leq 1$ then $\lim_{j\to\infty}(v_j) = 0$ as shown in Fig. 2a. However, given equation (15), $a > 1$ and so this scenario will never occur for our model.

If $1 < a \leq 3$ $\lim_{j\to\infty}(v_j) = 1 - (1/a)$ (as shown in Fig. 2b where $a = 2$), which from equation (16) implies that $\lim_{j\to\infty}(u_j) = U$. By setting $a = 3.2$ (Fig. 2c) we begin to observe the well known period doubling phenomena as a period-two orbit (one where the solution eventually repeats itself every second iteration) is produced. The next periodic orbit is of period 4, as illustrated in Fig. 2d where $a = 3.5$. By the time we reach $a = 4$ (Fig. 2e) the period has doubled an infinite number of times and hence the orbit never settles to a repeating pattern and we say that the orbit is chaotic. This infinite period orbit first occurs when the control parameter $a \simeq 3.5699$. Thus for $\gamma U \Delta t \geq 2.5699$ we can expect the model to produce chaotic results, but even for values of $1 < a \leq 3$, where the result settles down to $\lim_{j\to\infty}(u_j) = U$ it should be noted that the convergence is not necessarily monotonic.

4. Concluding remarks

Both of the models investigated here are unashamedly simple, neither do they ‘close the modelling loop’ by validating the model with real world data. However the aim of this paper has not been to illustrate the whole mathematical modelling process, rather it has been to concentrate on the aspects of formulation, solution and interpretation. In particular we have considered an example where the mode of solution (the Euler method) actually leads to the formulation of a new model. Further, equation (13) gives an introduction to the logistic map via traffic flow as an alternative to the more commonly used example of the Verhulst model of population growth [7]. In both cases the mathematics involved is straightforward, but the resulting solutions of the models leave room for the student to develop skills in manipulation and interpretation of formulae. As for the existence of genuine chaotic behaviour (in the mathematical sense of the word) on the roads, more complex car following models suggest that it may indeed exist [8].

5. Extensions and classroom exercises

(a) In the discussion of equation (10) the case of $\beta = 1$ is ignored. Investigate this case, remembering the result $e = \lim_{n\to\infty}(1 + (1/n))^n$. What happens if $\beta = -1$?

(b) The displcement at time t of the car from its starting position is given by the integral

$$x(t) = \int_0^t u(\omega)d\omega.$$

In the case of the Inattentive Driver model (4) this integral can be evaluated exactly for the following car by using the Trapezoidal rule to give the displacement of the following vehicle at time Δtj as,

$$x(\Delta tj) = \frac{1}{2}\Delta t(u_0 + u_j) + \Delta t \sum_{i=1}^{j-1} u_i.$$

By using equation (10) show that the distance, at time $t = n\Delta t$, between a following car which starts from rest and a lead car moving at constant velocity is given by

$$X(n\Delta t) = \frac{1}{2} U \Delta t \frac{1+\beta}{1-\beta}(1-\beta^n)$$

where the initial separation is taken to be $X(0) = 0$. Is the condition $X(n\Delta t) > 0 \forall n \in \mathbb{N}$ sufficient to ensure that $X(t) > 0 \forall t \in \mathbb{R}^+$? That is, is the fact that the vehicles have a positive separation at each observation time sufficient to ensure that the vehicles have a positive separation (and hence do not collide!) for all time?

(c) The models defined by equations (12) and (13) have a simple pathology associated with them for a particular value of following car velocity. What is it and how could the problem most simply be resolved? It is worth noting that this is a 'problem' for a whole class of car following models known as the Gazis–Herman–Rothery (or GHR) models. A survey of the GHR models can be found elsewhere [3].

(d) In the text it was noted that equation (14) may exhibit chaotic behaviour when $\gamma U \Delta t \geq 2.5699$. What sorts of driver velocities and values of Δt would be necessary for chaotic behaviour to occur?

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