Solutions of the unperturbed equation

Consider the unperturbed Burgers equation

$$w_t = 2\alpha\, w w_x + D w_{xx}$$

Subscripts denote derivatives. (D is the diffusion coefficient, – the viscosity in fluid dynamics.) With the Cole-Hopf transformation:

$$w = \frac{D}{\alpha}\frac{f_x}{f}$$

one obtains:

$$f\left(f_t - D f_{xx}\right)_x - f_x\left(f_t - D f_{xx}\right) = 0$$

The unperturbed Burgers equation is solved by f that obeys a linear diffusion equation:

$$f_t - D f_{xx} = 0 \quad (*)$$

This is what saying that the Burgers equation is "linearisable" means.

To get an idea about the necessary boundary conditions that need to be imposed, let us study the solutions of the equation (*). We are interested in wave solutions:

$$f(x,t) = F(x + V t)$$

where V is some fixed velocity. Denoting

$$\xi = x + V t$$

The equation (*) becomes

$$V f_\xi - D f_{\xi\xi} = 0$$

This equation is solved by

$$f = a + b\exp\left(\frac{V}{D}\xi\right) = a + b\exp\left(\frac{V}{D}x + \frac{V^2}{D}t\right)$$

This yields for the unperturbed Burgers equation the solution

$$w(x,t)=\frac{\dfrac{bV}{\alpha}\exp\left(\dfrac{V}{D}x+\dfrac{V^2}{D}t\right)}{a+b\exp\left(\dfrac{V}{D}x+\dfrac{V^2}{D}t\right)}$$

This is a "shock wave" or "front" solution. For V>0, one has at a fixed time

$$w(x,t)\to\begin{cases}0 & x\to-\infty\\ V/\alpha & x\to+\infty\end{cases}$$

$$w_x(x,t)\to 0 \qquad x\to\pm\infty \qquad (**)$$

The width of the transition region between the two asymptotic values of $w(x,t)$ is (D/V). (The analysis in Burgers' book is an expansion for small D (small viscosity)). The front propagates to the left at a speed $-V$. In the paper, they discuss solutions of this type.

Solution of the integrable perturbed equation

For simplicity, I now choose α=D=1. The following perturbed Burgers equation is integrable

$$w_t=2ww_x+w_{xx}+\varepsilon\lambda\left(3w^2w_x+3ww_{xx}+3w_x^{\ 2}+w_{xxx}\right)$$

because it is linearised by the same Cole-Hopf transformation, which now yields:

$$f\left(f_t-f_{xx}-\varepsilon\lambda f_{xxx}\right)_x-f_x\left(f_t-f_{xx}-\varepsilon\lambda f_{xxx}\right)=0$$

The "linearisable" perturbed Burgers equation is solved by f that obeys the linear equation:

$$f_t-f_{xx}-\varepsilon\lambda f_{xxx}=0$$

If we look for solutions of the type

$$f(x,t)=F(x+Vt) \quad , \qquad \xi=x+Vt$$

then we obtain for f the equation

$$Vf_\xi-f_{\xi\xi}-\varepsilon\lambda f_{\xi\xi\xi}=0$$

Looking for front solutions, we search for f of the form

$$f=\exp(\rho\xi)$$

and obtain for ρ

$$V\rho-\rho^2-\varepsilon\lambda\rho^3=0$$

The roots are

$$\rho = \begin{cases} 0 \\ \left(-1+\sqrt{1+4\varepsilon\lambda V}\right)/(2\varepsilon\lambda) = V\left[1-(\varepsilon\lambda V)+2(\varepsilon\lambda V)^2+O\left((\varepsilon\lambda V)^3\right)\right] \\ \left(-1-\sqrt{1+4\varepsilon\lambda V}\right)/(2\varepsilon\lambda) = -V\left[\dfrac{1}{(\varepsilon\lambda V)}+1-(\varepsilon\lambda V)+2(\varepsilon\lambda V)^2+O\left((\varepsilon\lambda V)^3\right)\right] \end{cases}$$

The solution for f is:

$$f = \begin{cases} a + b\exp\left[V\xi\left(1-(\varepsilon\lambda V)+2(\varepsilon\lambda V)^2+O\left((\varepsilon\lambda V)^3\right)\right)\right]+ \\ c\exp\left[-V\xi\left((1/(\varepsilon\lambda V))+1-(\varepsilon\lambda V)+2(\varepsilon\lambda V)^2+O\left((\varepsilon\lambda V)^3\right)\right)\right] \end{cases}$$

The term with $(1/\varepsilon)$ in the exponent generates a front the width of which is $O(\varepsilon)$. It may cause troubles if used in higher-order perturbation expansion. The analysis of the boundary conditions at $x \to \pm\infty$ follows the one presented earlier.

Analysis of the general perturbed equation

Consider now the general perturbed Burgers equation

$$u_t = 2uu_x + u_{xx} + \varepsilon\left(3\alpha_1 u^2 u_x + 3\alpha_2 uu_{xx} + 3\alpha_3 u_x{}^2 + \alpha_4 u_{xxx}\right)$$

The $O(\varepsilon)$ terms are of the types that appear in the expansion of the fluid dynamical equations that lead to the Burgers equation in lowest order.

The perturbed equation may not be "linearisable." We write a near-identity transformation (NIT):

$$u = w + \varepsilon\phi + \varepsilon^2\psi + \cdots$$

We assume no explicit dependence of ϕ on x or on t:

$$\phi = \phi\left(v, w, w_x, w_{xx}, \ldots\right) \qquad \text{where} \qquad v = \int_{x_0}^{x} w\,dx$$

The lower limit of integration, x_0, is arbitrary. If x_0 is not specified, the integral is defined up to a constant in x, which may depend on time. Let us choose $x_0 = -\infty$, so as to be able to impose boundary condition (**).

The normal form notation is the equation for the time dependence of the zero-order term::

$$w_t = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots$$

Inserting the NIT and the normal form in the original equation, we find in lowest order:

$$U_0 = 2w\,w_x + w_{xx}$$

Some useful identities:

$$U_{0,x} = 2w_x^{\,2} + 2w\,w_{xx} + w_{xxx}$$

$$\int_{x_0}^{x} U_0\,dx = w(x,t)^2 + w_x(x,t) - w(x_0,t)^2 - w_x(x_0,t)$$

Choosing $x_0=-\infty$, and zero-boundary condition at $x_0=-\infty$, the contribution of the lower limit to the integral vanishes.

Other useful identities:

$$v_t = \int_{-\infty}^{x} w_t\,dx = \int_{-\infty}^{x} \left(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \cdots\right) dx$$

$$\begin{aligned}\phi_t = {} & \phi_w\,w_t + \phi_{w_x}\,w_{tx} + \phi_v\,v_t = \\ & \phi_w\left(2\,w w_x + w_{xx} + \varepsilon\,U_1 + \cdots\right) + \\ & \phi_{w_x}\left(2\,w_x^{\,2} + 2\,w w_{xx} + w_{xxx} + \varepsilon\,U_{1,x} + \cdots\right) + \\ & \phi_v\left\{w^2 + w_x + \int_{-\infty}^{x}\left(\varepsilon U_1 + \cdots\right) dx\right\}\end{aligned}$$

(Note: in the last line of this identity the coefficient of the ϕ_v term had a contribution of the form

$$-\,w(x_0)^2 - w_x(x_0)$$

which I have already canceled, due to the choice $x_0=-\infty$. This will be consistently done in the following.)

$$\phi_x = \phi_w\,w_x + \phi_{w_x}\,w_{xx} + \phi_v\,w$$

$$\begin{aligned}\phi_{xx} = {} & \phi_{ww}\,w_x^{\,2} + \phi_{w_x w_x}\,w_{xx}^{\,2} + \phi_{vv}\,w^2 + \\ & 2\phi_{ww_x}\,w_x\,w_{xx} + 2\phi_{wv}\,w\,w_x + 2\phi_{w_x v}\,w\,w_{xx} + \\ & \phi_w\,w_{xx} + \phi_{w_x}\,w_{xxx} + \phi_v\,w_x\end{aligned}$$

Using all these relations, one obtains the following relation in $O(\varepsilon)$:

$$U_1 + \phi_w\left(\underline{2ww_x} + \underline{w_{xx}}\right) + \phi_{w_x}\left(2w_x^{\ 2} + \underline{2ww_{xx}} + \underline{w_{xxx}}\right) + \phi_v\left(w^2 + \underline{w_x}\right) =$$
$$\phi_w \underline{2ww_x} + \phi_{w_x} \underline{2ww_{xx}} + \phi_v 2w^2 + \phi 2w_x +$$
$$\phi_{ww} w_x^{\ 2} + \phi_{w_x w_x} w_{xx}^{\ 2} + \phi_{vv} w^2 +$$
$$2\phi_{ww_x} w_x w_{xx} + 2\phi_{wv} ww_x + 2\phi_{w_x v} ww_{xx} +$$
$$\phi_w \underline{w_{xx}} + \phi_{w_x} \underline{w_{xxx}} + \phi_v \underline{w_x} +$$
$$3\alpha_1 w^2 w_x + 3\alpha_2 ww_{xx} + 3\alpha_3 w_x^{\ 2} + \alpha_4 w_{xxx}$$

The underlined terms cancel out. What remains is:

$$U_1 = w^2\left\{\phi_v + \phi_{vv}\right\} + w_x^{\ 2}\left\{\phi_{ww} - 2\phi_{w_x} + 3\alpha_3\right\} + w_{xx}^{\ 2}\left\{\phi_{w_x w_x}\right\} +$$
$$ww_x\left\{2\phi_{wv}\right\} + ww_{xx}\left\{2\phi_{w_x v} + 3\alpha_2\right\} + w_x w_{xx}\left\{2\phi_{ww_x}\right\} + w^2 w_x\left\{3\alpha_1\right\} +$$
$$w_x\left\{2\phi\right\} + w_{xxx}\left\{\alpha_4\right\}$$

We treat this as a polynomial in w, w_{xx}, v, ... We want U_1 to be a "resonant" term, namely, a symmetry of the unperturbed equation. The definition of a resonant term is through the vanishing of its Lie-bracket with the unperturbed operator:

$$\left[2ww_x + w_{xx}, U_1\right] = 0$$

This gives for U_1 the form:

$$U_1 = \alpha_4\left(3w^2 w_x + 3ww_{xx} + 3w_x^{\ 2} + w_{xxx}\right)$$

This form guarantees that the normal form equation through first order is linearisable by the Cole-Hopf transformation, hence the normal form is integrable.

The remainder has to satisfy several identities. The first three obvious required identities are:

Coefficient of w_{xx}^2=0: $\phi_{w_x w_x} = 0$

Coefficient of $w_x w_{xx}$=0: $\phi_{ww_x} = 0$

Coefficient of ww_{xx}=$3\alpha_4$: $\phi_{w_x v} = \frac{3}{2}\left(\alpha_4 - \alpha_2\right)$

The first identity implies

$$\phi = A(w,v)w_x + B(w,v)$$

The second identity implies

$$A = A(v)$$

The third identity implies

$$A_v = \tfrac{3}{2}(\alpha_4 - \alpha_2) \Rightarrow A = \tfrac{3}{2}(\alpha_4 - \alpha_2)v + \alpha$$

where α is a free constant (which cannot be determined in this order). Hence,

$$\phi = \alpha w_x + \tfrac{3}{2}(\alpha_4 - \alpha_2) w_x v + B(w,v)$$

Inserting this result in the expression for U_1, we get

$$\begin{aligned} U_1 = {} & w^2\{B_v + B_{vv}\} + w_x{}^2\{B_{ww} + 3\alpha_3\} + \\ & w w_x\{2B_{wv}\} + w_x\{2B(w,v)\} + \\ & w^2 w_x\{3\alpha_1 + \tfrac{3}{2}(\alpha_4 - \alpha_2)\} + w w_{xx}\{3\alpha_4\} + w_{xxx}\{\alpha_4\} \end{aligned}$$

We have to kill the term

$$w^2\{B_v + B_{vv}\}$$

[No terms of the form (w^2 multiplied by functions of v) are desired in U_1.] Hence, we must have

$$B_v + B_{vv} = 0 \Rightarrow B = C(w)\exp(-v) + D(w)$$

This yields

$$\begin{aligned} U_1 = {} & w_x{}^2\{C_{ww}\exp(-v) + D_{ww} + 3\alpha_3\} + \\ & w w_x\{-2C_w\exp(-v)\} + w_x\{2C\exp(-v) + 2D\} + \\ & w^2 w_x\{3\alpha_1 + \tfrac{3}{2}(\alpha_4 - \alpha_2)\} + w w_{xx}\{3\alpha_4\} + w_{xxx}\{\alpha_4\} \end{aligned}$$

To get rid of the term proportional to

$$w_x \exp(-v)$$

(there are no $v-$dependent terms in U_1) we must have

$$-wC_w + C = 0 \qquad \Rightarrow \qquad C = \mu w$$

Here μ is a constant. This also eliminates the term

$$w_x{}^2 C_{ww}\exp(-v)$$

Hence, we obtain

$$U_1 = w_x{}^2\{D_{ww} + 3\alpha_3\} + w_x\{2D\} + \\ w^2 w_x\{3\alpha_1 + \tfrac{3}{2}(\alpha_4 - \alpha_2)\} + w\,w_{xx}\{3\alpha_4\} + w_{xxx}\{\alpha_4\}$$

The next identity is:

Coefficient of $w_x{}^2 = 3\alpha_4$: $\qquad D = \tfrac{3}{2}(\alpha_4 - \alpha_3)w^2 + k\,w + l$

where k and l are constants. To avoid a term linear in w_x, we must have $k=l=0$, yielding:

$$\phi = \alpha w_x + \mu w \exp(-v) + \tfrac{3}{2}(\alpha_4 - \alpha_2)\,w_x\,v + \tfrac{3}{2}(\alpha_4 - \alpha_3)w^2$$

$$U_1 = w^2 w_x\{3\alpha_1 + \tfrac{3}{2}(\alpha_4 - \alpha_2) + 3(\alpha_4 - \alpha_3)\} + \alpha_4\{3w_x{}^2 + 3w\,w_{xx} + w_{xxx}\}$$

We now have to take care of the remaining term:

$$w^2 w_x\{3\alpha_1 + \tfrac{3}{2}(\alpha_4 - \alpha_2) + 3(\alpha_4 - \alpha_3)\}$$

For its coefficient to be $3\alpha_4$, the relation

$$2\alpha_1 - \alpha_2 - 2\alpha_3 + \alpha_4 = 0 \qquad (***)$$

has to be satisfied. Otherwise, U_1 cannot have the "linearisable" form. One approach is to add the term, which cannot be eliminated in U_1, which then obtains the form:

$$U_1 = \underbrace{\alpha_4(3w^2 w_x + 3w\,w_{xx} + 3w_x{}^2 + w_{xxx})}_{\alpha_4 F_3(w)} + \underbrace{(3\alpha_1 - 3\alpha_3 - \tfrac{3}{2}\alpha_2 + \tfrac{3}{2}\alpha_4)w^2 w_x}_{Z(w)}$$

However, as now U_1 does not have the "linearisable" form, the normal form equation is not integrable, hence the notion of "obstacle" to integrability. Obstacles to integrability appear in higher orders as well. Another approach is to insist on the original idea of the normal form being comprised of resonant terms only, and to include the "obstacle" term in the equation for the first-order correction, ϕ. One conclusion is obvious: The assumption that the normal form expansion can be carried out under the assumption that all entities in the NIT and the normal form are differential polynomials in the zero-order term, w, does not work.

Comment

Look at the final form of ϕ:

$$\phi = \alpha w_x + \mu w \exp(-v) + \tfrac{3}{2}(\alpha_4 - \alpha_2)\,w_x\,v + \tfrac{3}{2}(\alpha_4 - \alpha_3)w^2$$

There are two arbitrary terms in ϕ:

$$\alpha w_x \quad \text{and} \quad \mu\, w \exp(-v)$$

The equations are transparent to them. The first one may be modified by varying the lower limit of integration in the definition of v, through the term:

$$\tfrac{3}{2}(\alpha_4 - \alpha_2) w_x v$$

The contribution of the lower limit of integration in the definition of v has the general form

$$-\tfrac{3}{2}(\alpha_4 - \alpha_2) v(x_0, t) w_x$$

which is linear in w_X. Thus, the choice of the boundary point can affect α. For $x_0=-\infty$ one has $v(x_0,t)=0$, thus, to de-coupling of α from the contribution of the lower limit.

Effect of obstacle on first-order term

If one solves the problem with a zero-order term which is a wave front with a single velocity (of the type discussed at the beginning), then one CAN solve for the first-order term, ϕ, because then the obstacle does not appear. One then finds:

$$\phi = (\alpha_1 - 2\alpha_2 - \alpha_3 - 2\alpha_4) w_x v - \tfrac{1}{2}(2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4) u^2$$

Now going to the general problem (where the zero-order solution is NOT a single-velocity wave), we assume that ϕ has a term, the form of which is like that obtained for a single-velocity wave solution + a correction, which we'll denote by χ:

$$\phi = (\alpha_1 - 2\alpha_2 - \alpha_3 - 2\alpha_4) w_x v - \tfrac{1}{2}(2\alpha_1 - \alpha_2 + \alpha_3 - 2\alpha_4) u^2 + \chi$$

The equation that χ obeys is:

$$\chi_t - 2 w_x \chi - 2 w \chi_x - \chi_{xx} = \Gamma \equiv (2\alpha_1 - \alpha_2 + 2\alpha_3 + \alpha_4)(w^2 w_x + w w_{xx} - w_x^{\,2})$$

Notice that the coefficient of the obstacle is just the quantity that needs to vanish, for there NOT to be an obstacle. However, the obstacle may vanish even if that coefficient does not. This happens in the case if w is a single-velocity wave. If w is not, then the obstacle may exist.

Let's look at the specific case of a two-velocity wave:

$$w = \frac{k_1 B \exp\left(k_1\left(x + \tilde{k}_1 t\right)\right) + k_2 C \exp\left(k_2\left(x + \tilde{k}_2 t\right)\right)}{1 + B \exp\left(k_1\left(x + \tilde{k}_1 t\right)\right) + C \exp\left(k_2\left(x + \tilde{k}_2 t\right)\right)} \qquad \left(\tilde{k}_i = k_i + \varepsilon \alpha_4 k_i^{\,2} \qquad i = 1,2\right)$$

Then the obstacle does not vanish. However, a close study of the obstacle, Γ, shows the following behaviour. For fixed x, when t is negative, Γ vanishes exponentially in time, as $t \to -\infty$, while for $t \to +\infty$, $\Gamma \to k_1\, k_2\, w_x$.

Thus, asymptotically, for large negative t, the obstacle vanishes, hence the correction χ vanishes exponentially in time. Asymptotically, for large positive t, Γ may be replaced by $w_{x,}$ converting the equation for χ into

$$\chi_t - 2\, w_x \chi - 2\, w\, \chi_x - \chi_{xx} \cong (2\,\alpha_1 - \alpha_2 + 2\,\alpha_3 + \alpha_4)\, k_1\, k_2\, w_x$$

This is obviously solved by

$$\chi = \tfrac{1}{2}(2\,\alpha_1 - \alpha_2 + 2\,\alpha_3 + \alpha_4)\, k_1\, k_2\ \chi = \tfrac{1}{2}(2\alpha_1 - \alpha_2 + 2\alpha_3 + \alpha_4)k_1\, k_2$$

Thus, asymptotically for large positive t, χ goes to a constant. The transition between the two constant asymptotic values is a function of t and x.