Weakly nonlinear oscillations: A perturbative approach

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The perturbative analysis of a one-dimensional harmonic oscillator subject to a small nonlinear perturbation is developed within the framework of two popular methods: normal forms and multiple time scales. The systems analyzed are the Duffing oscillator, an energy conserving oscillatory system, the cubically damped oscillator, a system that exhibits damped oscillations, and the Van der Pol oscillator, which represents limit-cycle systems. Special emphasis is given to the exploitation of the freedom inherent in the calculation of the higher-order terms in the expansion and to the comparison of the application of the two methods to the three systems.
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I. INTRODUCTION

An approximate linear model successfully describes many physical systems. However, nonlinear phenomena are encountered in all areas of the sciences. Almost always, nonlinear systems have fundamental aspects that cannot be faithfully treated by a linear approximation. Perhaps most important is the loss of the principle of superposition which plays a critical role in linear systems.

The behavior of nonlinear systems is much richer than that of linear systems. Unlike linear differential equations, the singularity structure of the solution of a nonlinear differential equation is affected by the initial conditions. As a result, a systematic classification of the solutions based on the structure of the differential equations is not generally possible.

In many cases, a simple closed form expression for the solution cannot be found, as is the case for chaotic systems, especially if their long-time behavior is sought. Even the analysis of nonchaotic systems does not lead to simple closed-form expressions for the solutions in most cases. Dynamical systems that can be solved by relatively simple means and to which a small perturbation has been added are the most manageable. In such situations, a perturbation expansion based on the solution of the unperturbed problem is used to obtain an approximate solution to the full problem. Historically, the first perturbed systems that were analyzed were those for which the dynamical variables are functions of time only. Moreover, these are systems for which the unperturbed equations are linear in the dynamical variables. Simple examples are harmonic oscillators and systems that decay exponentially.[1–17] The methods developed for the analysis of these systems were later applied to more complex dynamical systems of interest in fluid dynamics, plasma physics, reaction-diffusion systems, and population ecology.[18–26] In many of these latter systems, the dynamical variables depend both on time and position, so that they are governed by partial differential equations, making the analysis much more complicated. Because the purpose of this paper is to serve as an introductory tutorial for perturbation methods, we will focus on simpler systems that are described by a single dynamical variable, which is assumed to depend only on time (so that its governing equation is an ordinary differential equation) and for which the unperturbed equation is linear.

Numerous perturbation schemes have been developed for the analysis of systems that are affected by a small nonlinear perturbation. Each method is tailored to address the questions asked and the desired form of the answers. In this article, we follow two fundamentally different approaches. The first is the *method of normal forms*,[1–10,17] which has a firm mathematical foundation. Here one introduces a near identity transformation from the original set of variables to a new set whose equations are more suitable for analysis. The second method is an ad hoc perturbation scheme known as the *method of multiple time scales*.[11–16] It has a tenuous mathematical foundation, but has been applied with much more success to many types of equations, including perturbed oscillations, boundary-layer problems, and many partial differential equations.[23–26] The idea is to introduce independent time scales that characterize different aspects of the solution. We will show how the freedom in the perturbation expansion allows us to address the imposition of consistency conditions that arise in the expansion. In many cases, the methods of normal forms and multiple time scales are equivalent, and the method of choice depends on the application and on one's inclination. However, open problems exist in both methods, some of which will be discussed.

The O symbol is used frequently in the context of perturbation expansions. A quantity, g, that depends on a parameter, λ, is said to be $O(\lambda)$ if some positive constant, C, exists such that $|g(\lambda)| \leqslant C|\lambda|$.

We analyze three systems. Despite their simplicity, the analysis reveals much of what may be encountered in the application of the two methods to more complex systems. Some of the richness as well as the obstacles that perturbation theory generates are not evident to first order. Therefore, the analysis is carried at least through second order.

(1) The Duffing oscillator is defined by

$$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0, \quad 0 < \epsilon \ll 1, \tag{1}$$

where $x(0)=A$ and $dx/dt(0)=0$. Equation (1) provides an approximate model for anharmonic effects in crystal vibrations. For $\epsilon<0$, it can be viewed as a truncated version of the time dependence of the angular displacement, θ, of the pendulum:

$$L\frac{d^2\theta}{d\tau^2}+g\sin\theta=0, \tag{2}$$

where L is the length of the pendulum and g is the acceleration due to gravity. For small oscillation amplitudes, Eq. (2) has the same form as Eq. (1) if only the first two terms of the Taylor series expansion of $\sin\theta$ are kept. Although Eq. (2) is solved by elliptic integrals, we exploit its truncated version to show how approximate solutions are generated in an expansion scheme.

For an understanding of perturbative expansion methods, there is no gain in including higher-order terms in the Taylor series in Eq. (1).[11,17] Because Eq. (1) describes a conservative system, the trajectory in the phase space formed by x and dx/dt *is* a closed curve. Note that ϵA^2, which is a measure of the perturbation to the energy of the system, is the true small parameter, not ϵ.

(2) A linear oscillator with a small cubic damping perturbation:

$$\frac{d^2x}{dt^2}+x+\epsilon\left\{\frac{dx}{dt}\right\}^3=0 \tag{3}$$

where $0<\epsilon\ll 1$. We discuss cubic damping for two reasons. First, linear damping does not generate the richness of phenomena generated by nonlinear perturbations. Second, the interesting effects of quadratic damping appear only in $O(\epsilon^2)$, whereas they emerge in the first-order analysis of the cubic damping case.[11,17] Hence, the latter is a good example for illustrating features of perturbation theory.

(3) The Van der Pol oscillator:[27]

$$\frac{d^2x}{dt^2}+x=\epsilon\dot{x}(1-x^2), \tag{4}$$

where $0<\epsilon\ll 1$. Equation (4) was introduced to model an electric circuit that contains a nonlinear component called a triode, and is equivalent to that developed by Lord Rayleigh to model self-sustained sound vibrations.[28] The solution of Eq. (4) evolves into a limit cycle, an aspect of nonlinear oscillations that has no parallel in linear oscillatory system, and occurs in many dissipative nonlinear systems, such as the onset of convective rolls in a fluid.[23] A limit cycle is a closed orbit in phase space that is approached asymptotically from nearby orbits. For small amplitude, x, the right-hand side of Eq. (4) has the same sign as the velocity, and represents a force that pumps energy into the system, causing the amplitude to increase. As the displacement exceeds unity, the dissipative term changes sign, leading to deceleration, and, eventually, a halt to amplitude growth. The amplitude then decreases and falls below unity, where, once again, the dissipative term feeds energy into the system, leading to a repeat of the behavior. For $t\to\infty$, the solution tends to the limit cycle. The zero-order approximation to the solution is called the limit circle. The limit cycle is the limit circle modified by higher-order corrections.

II. INTRODUCING THE STANDARD FORM

We first convert our second-order differential equations into two-dimensional vector equations:

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\epsilon\begin{pmatrix}0\\F(x,y)\end{pmatrix}, \tag{5}$$

where $y=dx/dt$ and

$$F=\begin{cases}-x^3 & \text{(Duffing oscillator)}\\ -\dot{x}^3 & \text{(Damped oscillator)}\\ \dot{x}(1-x^2) & \text{(Van der Pol oscillator)}\end{cases}. \tag{6}$$

It is not essential, but it is traditional to diagonalize the matrix associated with the linear problem. The diagonalization is accomplished by introducing $z=x+iy$ and its complex conjugate, $z^*=x-iy$. Equation (5) becomes

$$\frac{dz}{dt}=-iz+i\epsilon F(z,z^*). \tag{7}$$

III. THE METHOD OF NORMAL FORMS

We first introduce the near-identity transformation and write the complex variable z in terms of the zero-order approximation, z_0:

$$z=z_0+\sum_{n\geqslant 1}\epsilon^n z_n(z_0,z_0^*),\quad z^*=z_0^*+\sum_n\epsilon^n z_n^*(z_0,z_0^*). \tag{8}$$

We need only solve the z equation as the equation for z^* is its complex conjugate. The dynamical equation obeyed by z_0, the normal form, is assumed to have the form

$$\frac{dz_0}{dt}=U_0+\sum_{n\geqslant 1}\epsilon^n U_n(z_0,z_0^*), \tag{9}$$

where $U_0=-iz_0$.

The following points provide some perspective and motivation for the development of the method.

(1) The zero-order approximation, z_0, need not be restricted to obey the unperturbed equation. Rather, it is assumed to be affected by the perturbation parameter, ϵ.

(2) The quantity U_0 is the unperturbed operator; the higher-order terms in the normal form, U_n, update the time dependence of z_0 due to the presence of the perturbation.

(3) The higher-order terms in the expansion of the solution, z_n, depend on z_0 and z_0^*.

(4) The key role of the near-identity transformation is to exploit z_n so that as many nonlinear terms are included as possible, yet ensuring that the normal form is solvable and incorporates the major effect of the perturbation already in z_0.

(5) An obvious choice for the structure of z_n is to mimic the form of the perturbation, leading to a set of linear equations for the desired coefficients. With the normal form solved for z_0, the z_n are then found and an approximate solution is obtained to the desired order of the expansion.

(6) Central to our approach is the exploitation of the freedom in the structure of the z_n (see Refs. 29–38) to modify aspects of the expansion or to satisfy desired constraints.[17]

(7) The method of normal forms provides an approximation to the exact (but unknown) solution, $z(t)$. The error,

which is the deviation between the exact and approximate solutions, remains bounded for a limited time span. If we truncate the near-identity transformation at a finite order, thus generating an approximation $z_a(t)$,

$$z_a = z_0 + \sum_{n=1}^{N} \epsilon^n z_n(z_0, z_0^*), \tag{10}$$

and, concurrently, the normal form, Eq. (9) is calculated through order $N+1$ (that is, one order beyond that included in the near-identity transformation), then it can be proven that the error incurred is

$$|z(t) - z_a(t)| = O(\epsilon^{N+1}), \tag{11}$$

for a time span that is at most of order $(1/\epsilon)$ [the common notation is $t = O(1/\epsilon)$].

Rather then going through the detailed proof (see, for example, Ref. 17), we provide an intuitive explanation. The approximation, $z_a(t)$, is computed through $O(\epsilon^N)$. Thus, seemingly, the error ought to be $O(\epsilon^{N+1})$. However, the approximate time dependence, even of the zero-order term, is an independent source of error. To be specific, suppose that $z_0(t)$ is proportional to $\cos \omega t$. If the correct value of ω is replaced by an approximate value, ω_a, then

$$|\cos \omega t - \cos \omega_a t| = 2\left|\sin\left(\frac{\omega+\omega_a}{2}t\right)\right|\left|\sin\left(\frac{\omega-\omega_a}{2}t\right)\right|. \tag{12}$$

If the normal form has been computed through $O(\epsilon^{N+1})$, the method yields $(\omega - \omega_a) = O(\epsilon^{N+2})$. If t is not too large, the second sine function in Eq. (12) can be approximated by its argument, which is $O(\epsilon^{N+2}t)$, and, hence, the error is $O(\epsilon^{N+1})$ only for $t = O(1/\epsilon)$.

IV. APPLICATIONS OF THE METHOD OF NORMAL FORMS

A. The Duffing oscillator

In terms of the complex variables z and z^*, we can write Eq. (1) as

$$\dot{z} = -iz - i\epsilon\frac{(z+z^*)^3}{8}. \tag{13}$$

We begin by demonstrating why the method of normal forms is necessary. Suppose that we ignore the normal form, Eq. (9), and substitute the near-identity transformation, Eq. (8), in Eq. (13). This procedure is called the naive expansion. Because a problem already will arise in first-order, we write the result only through $O(\epsilon)$:

$$\dot{z}_0 + \epsilon\dot{z}_1 = -i(z_0 + \epsilon z_1) - i\epsilon\frac{(z_0+z_0^*)^3}{8}. \tag{14}$$

We compare terms order by order. In $O(\epsilon^0)$ we have

$$\dot{z}_0 + iz_0 = 0, \tag{15}$$

which implies that $z_0 = \rho e^{-it}$. In $O(\epsilon^1)$ we have

$$\dot{z}_1 + iz_1 = -i\epsilon\rho^3\frac{(e^{-3it} + 3e^{-it} + 3e^{it} + e^{3it})}{8} \tag{16}$$

and thus

$$z_1 = \rho^3\left\{\frac{1}{16}e^{-3it} + \frac{3}{8}te^{-it} - \frac{3}{16}e^{it} - \frac{1}{32}e^{3it} + \alpha e^{-it}\right\}. \tag{17}$$

The last term in the solution in Eq. (17) is the general solution of the homogeneous part of Eq. (16); its coefficient is undetermined. The second term on the right-hand side of Eq. (16) resonates with the differential operator, $(\dot{z}_1 + iz_1)$, and hence is called a resonant term. It generates a contribution to z_1 that is linear in t Eq. (17). This contribution, called a secular term, limits the validity of the expansion to short times of $O(1)$. The reason is that the contribution of the first-order correction to the solution, ϵz_1, contains a term that is proportional to ϵt. Hence, for ϵz_1 to be of magnitude ϵ relative to the zero-order approximation, z_0, t must be $O(1)$. As t grows to $O(1/\epsilon)$, this term spoils the validity of the perturbative expansion. Secular terms also appear in the higher orders.

Terms that are linear in t also violate a requirement that is based on physical intuition. The original system is conservative. Hence, its solution must be a periodic function of time. This behavior does not mean that the perturbative approximation must be periodic in time as well, but such behavior is preferable. The avoidance of secular terms and the construction of the approximate solution from only periodic functions work hand in hand and guarantee that the expansion remains valid for longer times, of $O(1/\epsilon)$. As indicated by the discussion involving Eqs. (10)–(12), the error incurred by approximating the full solution by a first-order approximation is

$$|z(t) - \{z_0(t) + \epsilon z_1(z_0, z_0^*)\}| = O(\epsilon) \tag{18}$$

for $t = O(1/\epsilon)$. If we substitute Eqs. (8) and (9) in the left-hand side of Eq. (13) and expand the result through $O(\epsilon^2)$, we obtain:

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz_0}{dt} + \epsilon\left(\frac{\partial z_1}{\partial z_0}\frac{dz_0}{dt} + \frac{\partial z_1}{\partial z_0^*}\frac{dz_0^*}{dt}\right) \\ &\quad + \epsilon^2\left(\frac{\partial z_2}{\partial z_0}\frac{dz_0}{dt} + \frac{\partial z_2}{\partial z_0^*}\frac{dz_0^*}{dt}\right) \\ &= \{U_0 + \epsilon U_1 + \epsilon^2 U_2\} + \epsilon\left(\frac{\partial z_1}{\partial z_0}U_0 + \frac{\partial z_1}{\partial z_0^*}U_0^*\right) \\ &\quad + \epsilon^2\left(\frac{\partial z_2}{\partial z_0}U_0 + \frac{\partial z_2}{\partial z_0^*}U_0^*\right) + \epsilon^2\left(\frac{\partial z_1}{\partial z_0}U_1 + \frac{\partial z_1}{\partial z_0^*}U_1^*\right).\end{aligned} \tag{19}$$

Next we substitute the expansion for z given by Eq. (8) into the right-hand side of Eq. (13):

$$\begin{aligned}\frac{dz}{dt} &= -i\{z_0 + \epsilon z_1 + \epsilon^2 z_2\} - i\epsilon\frac{(z_0+z_0^*)^3}{8} \\ &\quad - i\epsilon^2\frac{3(z_0+z_0^*)^2}{8}(z_1 + z_1^*).\end{aligned} \tag{20}$$

We equate Eqs. (19) and (20), collect the ϵ^0 terms, and find

$$U_0 = -iz_0, \tag{21}$$

which is the eigenvalue equation for the unperturbed problem. We then solve for each order in the small parameter ϵ, a series of equations for the U_i. To $O(\epsilon)$, we have:

$$U_1=\left\{-iz_1-U_0\frac{\partial z_1}{\partial z_0}-U_0^*\frac{\partial z_1}{\partial z_0^*}\right\}-i\frac{1}{8}(z_0+z_0^*)^3. \quad (22)$$

The structure of the term in curly braces in Eq. (22) is independent of the interaction. It is known as the Lie bracket for this problem and in every order has the form:

$$[U_0,z_n]\equiv\left\{-iz_n-U_0\frac{\partial z_n}{\partial z_0}-U_0^*\frac{\partial z_n}{\partial z_0^*}\right\}. \quad (23)$$

Because the perturbation is a polynomial in z_0 and z_0^*, we expect z_n to have a form that mimics this polynomial. The calculation leads to the result that the nth-order term, z_n, is a polynomial of degree $(2n+1)$ in z_0 and z_0^*. Thus, to first order we write

$$z_1=\sum_{\substack{k,m\geqslant 0\\ k+m=3}}\alpha_{km}z_0^k z_0^{*m}. \quad (24)$$

If we substitute Eq. (24) into Eq. (23), we observe that the Lie bracket takes the form:

$$[U_0,z_1]=-i\sum_{\substack{k,m\geqslant 0\\ k+m=3}}\alpha_{km}z_0^k z_0^{*m}[1-k+m]. \quad (25)$$

We choose the α coefficients to eliminate as many as possible of the terms that were generated by the perturbation on the right-hand side of Eq. (22). Only the resonant term, which is proportional to $z_0^2z_0^*$, cannot be eliminated. The reason is that because α_{21} is multiplied by zero in Eq. (25), it is undetermined. (In future expressions, we indicate this free multiplier as α.) Because the $z_0^2z_0^*$ term is responsible for the appearance of the secular contribution in Eq. (17), it has to be removed from the equation that determines z_1 by assigning it to the normal form. This term determines the structure of U_1. We use Eqs. (20)–(25) and solve for z_1 to find

$$z_1=\tfrac{1}{16}z_0^3-\tfrac{3}{16}z_0z_0^{*2}-\tfrac{1}{32}z_0^{*3}+\alpha z_0^2z_0^*, \quad (26)$$

and

$$U_1=-i\tfrac{3}{8}z_0^2z_0^*. \quad (27)$$

Note that, were we to include in z_1 a resonant term of the form $z_0^{n+1}z_0^{*n}$ for any $n\geqslant 1$, Eq. (25) would not determine its coefficient, $\alpha_{n+1,n}$. Hence, the most general form of the resonant term is

$$\sum_n \alpha_{n+1,n}z_0^{n+1}z_0^{*n}=f(z_0\cdot z_0^*)z_0. \quad (28)$$

For a conservative system with a single degree of freedom, this complication turns out to be unnecessary, and we stick to the resonant term as written in Eq. (26). In more complicated systems (in particular, dissipative ones) the more general from of the resonant terms may be needed.

The calculation of the second-order correction follows the same pattern, and we obtain

$$z_2=\tfrac{3}{1024}z_0+(-\tfrac{15}{256}+\tfrac{3}{16}\alpha)z_0^4z_0^*+(\tfrac{69}{512}-\tfrac{3}{16}\alpha-\tfrac{3}{8}\alpha^*)z_0^2z_0^{*3}$$
$$+(\tfrac{21}{1024}-\tfrac{3}{32}\alpha^*)z_0z_0^{*4}-\tfrac{1}{512}z_0^{*5}+\beta z_0^3z_0^{*2}, \quad (29)$$

and

$$U_2=i(\tfrac{51}{256}-\tfrac{3}{8}(\alpha+\alpha^*))z_0^3z_0^{*2}. \quad (30)$$

Again, all the resonant terms, which have the capacity to generate secular behavior, have been assigned to U_2, the second-order term in the normal form; the remaining terms have been used to construct z_2. The same pattern recurs in higher orders. In nth order, the formalism generates polynomials of degree $2n+1$ in z_0 and z_0^*. The resonant term, $z_0^{n+1}z_0^{*n}$, which would generate a secular term in z_n in the naive expansion, is assigned to U_n, the nth-order term in the normal form. Thus, in the normal form expansion, the nonresonant and the resonant terms that are generated by the perturbation play different roles. We construct z_n, the nth-order correction in the near identity transformation, Eq. (8), so that it accounts for all the nonresonant terms. The normal form, Eq. (9), is constructed from all the resonant terms and determines the time dependence of z_0, the zero-order approximation.

We note that the numerical coefficients in U_n are pure imaginary for all n, leading to real coefficients for all the z_n. [We have shown this explicitly through $O(\epsilon^2)$, but one can prove it for all orders by induction.] Therefore, the normal form has the following simple structure:

$$\dot{z}_0=-iz_0g(z_0z_0^*;\epsilon), \quad (31)$$

where g is a real function. If we write,

$$z_0=\rho\exp(-i\theta), \quad (32)$$

it is easy to show that the amplitude, ρ, is constant:

$$\frac{d\rho}{dt}=0. \quad (33)$$

Hence, the U_n affect only the frequency, ω, which is changed in each order of the expansion. We use the expressions for U_n, $n=1,2$, to obtain

$$\frac{d\theta}{dt}=\omega=1+\epsilon\frac{3}{8}\rho^2-\epsilon^2\left\{\frac{51}{256}-\frac{3}{8}(\alpha+\alpha^*)\right\}\rho^4. \quad (34)$$

As a result, the zero-order approximation, z_0, is given by

$$z_0=\rho\exp(-i\omega t)$$
$$=\rho\exp(-i\omega_0 t)\exp(-i\epsilon\omega_1 t)\exp(-i\epsilon^2\omega_2 t), \quad (35)$$

where

$$\omega=\omega_0+\epsilon\omega_1+\epsilon^2\omega_2,\quad \omega_0=1,\quad \omega_1=\tfrac{3}{8}\rho^2,$$
$$\omega_2=-\{\tfrac{51}{256}-\tfrac{3}{8}(\alpha+\alpha^*)\}\rho^4. \quad (36)$$

If we use our expressions for z_1 and z_2, we can now find z, z^*, and hence $x(t)$ through $O(\epsilon^2)$:

$$x(t)=\rho[1+\epsilon(-\tfrac{3}{16}+\alpha)\rho^2+\epsilon^2(\tfrac{69}{512}-\tfrac{9}{16}\alpha+\beta)\rho^4]\cos\theta$$
$$+\rho^3(\epsilon\tfrac{1}{32}+\epsilon^2(-\tfrac{39}{1024}+\tfrac{3}{32}\alpha)\rho^2)\cos 3\theta$$
$$+\epsilon^2\rho^5(\tfrac{1}{1024})\cos 5\theta. \quad (37)$$

Choice of free terms. The free terms that emerge in the analysis are an inherent feature of any perturbation theory. The freedom in their choice is a reflection of the fact that one does not aim at the *exact* solution, or at a precisely specified approximation. Rather, one searches for an approximation that is required to be close to the correct solution within a certain error range. Technically, this results in the fact that, in a given order, n, of the expansion, it is impossible to distinguish between the effect of the perturbation on the time dependence of the zero-order term and on the correction term of order n. The numerical quality of the approximation is not affected by different choices of the free terms. [For example, they cannot improve the error estimate from, say, $O(\epsilon)$ to $O(\epsilon^2)$ or vice versa.] We may choose the free terms in the expansion [proportional to α and β in Eqs. (34)–(37)] to satisfy a convenient requirement. Figure 1 provides a numerical example of the phase-plane plot of the full solution for the Duffing equation, together with a plot of the zero-order approximation, obtained through a first-order analysis with $\alpha=0$. A relatively large value of ϵ ($\epsilon=0.3$) was chosen to emphasize the effect of the perturbation on distorting the zero-order circle.

One place where the choice of the free functions can be useful is in the calculation of the period of the oscillations, which is given by an integral over one cycle in phase space:

$$T=\oint \frac{dx}{\sqrt{2(E-V(x))}}. \tag{38}$$

Here E is the total energy and $V(x)$ is the potential. If we substitute the potential and the total energy for the case of the Duffing oscillator,

$$V(x)=\tfrac{1}{2}x^2+\tfrac{1}{4}x^4, \quad E=\tfrac{1}{2}A^2+\tfrac{1}{4}A^4, \tag{39}$$

where A is the amplitude at the turning point, the integral becomes

$$T=4\int_0^A \frac{dx}{\sqrt{(A^2-x^2)+\dfrac{1}{2}\epsilon(A^4-x^4)}}. \tag{40}$$

This integral can be expressed in terms of a complete elliptic integral of the first kind. If we evaluate the integral through $O(\epsilon^4)$, we find an expression for the frequency $\omega=(2\pi/T)$ in terms of the A:

$$\omega=1+\epsilon\frac{3}{8}A^2-\epsilon^2\frac{21}{256}A^4+\epsilon^3\frac{81}{2048}A^6-\epsilon^4\frac{6549}{262\,144}A^8 +O(\epsilon^5). \tag{41}$$

In Eq. (36), ω is expressed in terms of ρ, the amplitude of the zero-order approximation. To properly compare Eqs. (36) and (41), we need to find the relation between ρ and A. We obtain this relation by imposing the initial conditions, $x(0)$ and $dx/dt(0)$. The choice of the free functions affects this relation as well as the manner in which the phase is changed.

Problem 1. Explain why the period can be calculated as four times the integral over a quarter of the cycle and derive Eq. (40).

The choice of the free functions (the constants α and β in the present problem) affects the structure of the higher-order correction terms in the near-identity transformation. As the preceding analysis shows, this choice modifies the structure of the zero-order approximation. Thus, different choices of the free functions correspond to different ways of arranging the perturbation series and shifting higher-order terms between the near-identity transformation, Eq. (8), and the normal form, Eq. (9). Clearly, the physics of the solution cannot change, and, indeed, a detailed check[17] shows that, through a given order in ϵ, the error incurred by using the approximate solution rather than the exact one is of the same quality for different choices of the free functions. For instance, if the near-identity transformation is carried through $O(\epsilon)$ and the normal form through $O(\epsilon^2)$, then, as Eq. (11) indicates, the error incurred is $O(\epsilon)$, namely, the error is bounded by $C\epsilon$, through times of $O(1/\epsilon)$ for some $C>0$. Although this statement does not depend on the choice of the free terms, the constant, C, may depend on the choice.

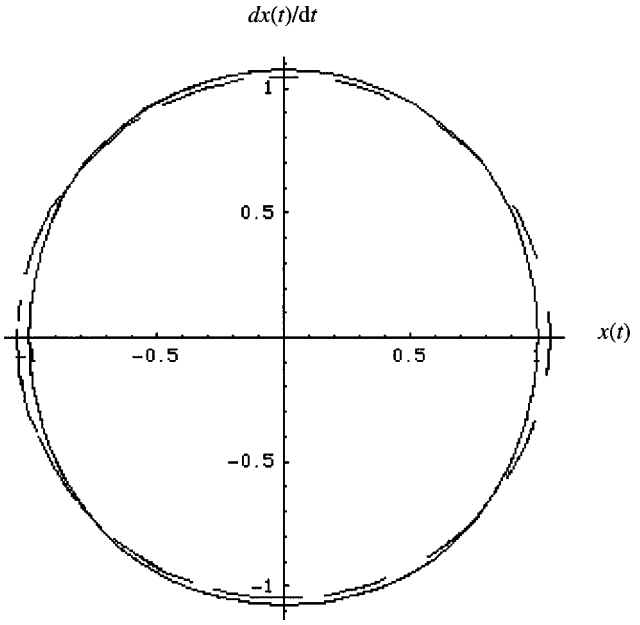

Fig. 1. Phase-plane plot of the solution for the Duffing oscillator with $\epsilon=0.3$, $x(0)=1$, and $dx/dt(0)=0$. Full solution (line) and zero-order circle approximation (dashed line) with $\alpha=0$ in Eq. (4.12).

Of special interest is the *minimal normal form* choice.[17,34–38] If we choose $\alpha=17/64$, then Eqs. (30) and (36) yield $U_2=\omega_2=0$. One can show by induction that U_n can be made to vanish for all $n\geqslant 2$. For instance, $\beta=131/8192$ makes U_3 and ω_3 vanish. The numerical approximations generated by the minimal normal form choice are better than all other choices. This fact is demonstrated in Fig. 2, where the error [the difference between the full numerical solution of Eq. (1) and the first-order approximation] is shown for the choice $\alpha=0$ and the minimal normal form choice of $\alpha=17/64$.

The formal truncation of the normal form is an example of the concept of *renormalization*. The origin of the ability to formally truncate the normal form at a finite order is the same as for the multipole expansion of electromagnetic potentials. The functional form of the first nonzero multipole is physically significant, whereas the functional form of all the higher multipoles depends on the choice of the origin of the axes in the frame of reference in which the expansion is performed. Similarly here, the first nontrivial correction to the frequency (a first-order one in the case of the Duffing oscillator) is physically significant, whereas all the higher-order corrections depend on the choice of the zero-order ap-

proximation (the analog of the choice of the origin in the multipole expansion). The latter is intimately related to the choice of the free terms in each order for the following reason. A resonant term is proportional to $z_0^{n+1}z_0^{*n}=(z_0z_0^*)^n z_0$. Because for a conservative system the amplitude, ρ, is constant [see, Eq. (33)], the free resonant terms that are included in the higher-correction terms z_n are proportional to the zero-order term, z_0. Thus, inclusion of free resonant terms in z_n amounts to shifting parts of the contribution of the zero-order term to higher orders.

B. The oscillator with cubic damping

The main features of the solution for the Duffing oscillator are typical of conservative systems with one degree of freedom: the amplitude, ρ, of the zero-order term, z_0, is constant, and the frequency is changed by constant terms, so that the phase is just ωt, with ω a constant that differs from the unperturbed frequency. The structure of the solutions in dissipative systems is different. Both the amplitude and the phase are complicated functions of time. For the oscillator with cubic damping, the solution is attracted to the origin due to damped oscillations. The calculations parallel those of the Duffing oscillator. If we use Eqs. (3), (6), and (7), we can write

$$\dot{z}=-iz-i\frac{(z-z^*)^3}{2i}=-iz+i\frac{(z-z^*)^3}{2}. \tag{42}$$

An example of the solution for $x(t)$, which is the real part of $z(t)$, is given in Fig. 3.

We substitute the near-identity transformation, Eq. (8), and the normal form, Eq. (9), in Eq. (42) and obtain:

$$\begin{aligned}\frac{dz}{dt}&=\{U_0+\epsilon U_1+\epsilon^2U_2\}+\epsilon\left(\frac{\partial z_1}{\partial z_0}U_0+\frac{\partial z_1}{\partial z_0^*}U_0^*\right)\\&\quad+\epsilon^2\left(\frac{\partial z_2}{\partial z_0}U_0+\frac{\partial z_2}{\partial z_0^*}U_0^*\right)+\epsilon^2\left(\frac{\partial z_1}{\partial z_0}U_1+\frac{\partial z_1}{\partial z_0^*}U_1^*\right)\\&=-i\{z_0+\epsilon z_1+\epsilon^2z_2\}+\epsilon\frac{1}{8}(z_0-z_0^*)^3\\&\quad+\epsilon^2\frac{3}{8}(z_0-z_0^*)^2(z_1+z_1^*).\end{aligned} \tag{43}$$

We collect terms as before and obtain:

$$U_0=-iz_0, \tag{44}$$

$$U_1=\left\{-iz_1-U_0\frac{\partial z_1}{\partial z_0}-U_0^*\frac{\partial z_1}{\partial z_0^*}\right\}+\frac{(z_0-z_0^*)^3}{8}. \tag{45}$$

If we use the expression in Eq. (24) for z_1, the $O(\epsilon)$ part of the solution is

$$U_1=-\epsilon\tfrac{3}{8}z_0^2z_0^*, \tag{46}$$

and

$$z_1=\tfrac{1}{16}iz_0^3-\frac{3}{16}iz_0z_0^{*2}+\tfrac{1}{32}iz_0^{*3}+f_1(z_0\cdot z_0^*)z_0. \tag{47}$$

The last term in z_1 is the free resonant term. Observe that z_1 has imaginary coefficients and that U_1 is real, indicating damped motion. In a similar manner, the $O(\epsilon^2)$ terms are found to be

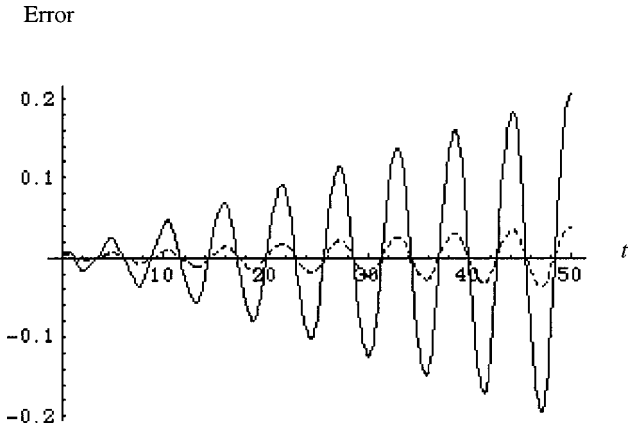

Fig. 2. Error in the first-order approximation to the solution for the Duffing oscillator with the same conditions as in Fig. 1. No free function (line) and minimal normal form (dashed line) choice.

$$U_2=-\tfrac{3}{8}(f_1+f_1^*)z_0^2z_0^*+\tfrac{3}{4}f_1'z_0^3z_0^{*2}+\tfrac{27}{256}iz_0^3z_0^{*2}, \tag{48}$$

$$\begin{aligned}z_2&=-\tfrac{9}{1024}z_0^5+\tfrac{9}{256}z_0^4z_0^*-\tfrac{27}{512}z_0^2z_0^{*3}-\tfrac{27}{1024}z_0z_0^{*4}+\tfrac{3}{512}z_0^{*5}\\&\quad+\tfrac{3}{16}if_1z_0^3-\tfrac{3}{16}if_1z_0^2z_0^{*2}-\tfrac{3}{8}if_1^*z_0z_0^{*2}+\tfrac{3}{32}if_1^*z_0^{*2}\\&\quad+f_2(z_0\cdot z_0^*)z_0,\end{aligned} \tag{49}$$

where

$$f_1'=\frac{d}{d(z_0z_0^*)}f_1(z_0z_0^*), \tag{50}$$

and a free resonant term has been included in z_2. To $O(\epsilon^3)$ we find

$$\begin{aligned}U_3&=-\tfrac{3}{8}(2f_1^2+3f_1f_1^*)z_0^2z_0^*+[\tfrac{27}{128}i(f_1+f_1^*)\\&\quad-\tfrac{3}{4}f_1f_1']z_0^3z_0^{*2}-\tfrac{3}{4}(f_1'+f_1'^*)f_1'z_0^4z_0^{*3}\\&\quad-\tfrac{3}{8}(f_2+f_2^*)z_0^2z_0^*+\tfrac{3}{4}f_2'z_0^3z_0^{*2}-\tfrac{567}{8192}z_0^4z_0^{*3}.\end{aligned} \tag{51}$$

We now show how the choice of the free resonant terms in the generators, z_n, may be exploited. If all the free terms vanish ($f_1=f_2=0$ in the approximation presented here), then the normal form is given by

$$\begin{aligned}\frac{dz_0}{dt}&=-iz_0-\epsilon\frac{3}{8}z_0^2z_0^*+\epsilon^2\frac{27}{256}iz_0^3z_0^{*2}-\epsilon^3\frac{567}{8192}z_0^4z_0^{*3}\\&\quad+O(\epsilon^4).\end{aligned} \tag{52}$$

If we use Eq. (32) for z_0, we find that Eq. (52) leads to the conclusion that is typical of dissipative systems equations, namely, that both the frequency, ω, and the amplitude, ρ, vary with time in a nontrivial manner:

$$\frac{d\rho}{dt}=-\frac{3}{8}\epsilon\rho^3-\frac{567}{8192}\epsilon^3\rho^7+O(\epsilon^4), \tag{53}$$

$$\frac{d\theta}{dt}=\omega=1-\frac{27}{256}\epsilon^2\rho^4+O(\epsilon^4). \tag{54}$$

In higher orders, the amplitude and phase equations quickly become unwieldy. However, in the present problem, the free functions can be chosen to eliminate U_n for all $n\geqslant 2$. For

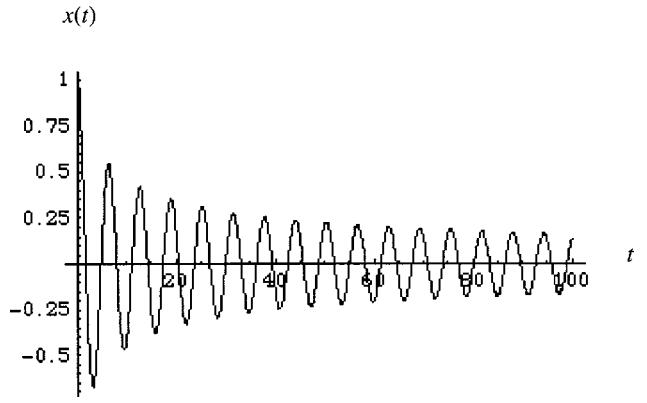

Fig. 3. The position $x(t)$ for an harmonic oscillator with cubic damping with the same conditions as in Fig. 1.

instance, requiring that U_2 vanish can be achieved for a variety of choices for f_1, the simplest one being

$$f_1 = -\tfrac{9}{64} i z_0 z_0^* . \tag{55}$$

In a similar manner, the simplest choice for f_2 that eliminates U_3 is

$$f_2 = \frac{675}{8192}(z_0 z_0^*)^2. \tag{56}$$

One can show by induction that higher-order resonant contributions to the normal form may be eliminated order-by-order to yield the following expression, correct to all orders, for the minimal normal form:

$$\frac{dz_0}{dt} = -iz_0 - \frac{3}{8}\epsilon z_0^2 z_0^* \tag{57}$$

which implies that

$$\frac{d\rho}{dt} = -\frac{3}{8}\epsilon\rho^3, \quad \frac{d\theta}{dt} = \omega = 1. \tag{58}$$

The solution of Eq. (58) is

$$\theta = t + \theta_0, \quad \Rightarrow \omega = 1, \quad \rho = \frac{\rho_0}{\sqrt{1 + \frac{3}{4}\epsilon\rho_0^2 t}}. \tag{59}$$

Thus, the minimal normal form captures the behavior of the zero-order approximation in a very simple manner. Both the fundamental frequency of oscillation and the amplitude are determined at this order and are not changed by higher-order corrections. Higher-order effects show up only in z_n in the near-identity transformation.

Problem 2. Solve Eqs. (53) and (54) numerically for the initial conditions $\rho(0)=1$ and $\theta(0)=0$. What do you learn about the phase θ from the solution? In what way does it differ from the phase for the Duffing oscillator?

Problem 3. Show that the choice, Eq. (55), makes U_2 vanish.

We note the following points. (1) The z_n have the same structure as those obtained for the Duffing oscillator. (2) The minimal normal form choice, so simple in this case, may not be a viable option in most dissipative systems. (The Van der Pol oscillator is an example.) (3) We note that, in first order, the coefficient of U_1 is real [see Eq. (46)], so that only ρ, the amplitude of z_0, is affected by the perturbation. In second order, for the common choice of $f_1 = f_2 = 0$, U_2 has an imaginary coefficient [see Eq. (48)], so that U_2 affects only the phase and not the amplitude of z_0. This pattern persists in all higher orders. The amplitude equation is changed only in odd orders in ϵ, and the phase equation only in even orders. (4) For conservative systems, the normal form expansion yields a phase that is linear in t because the frequency is changed by successive constant additive corrections. In dissipative systems, the correction terms are not constant, and the phase becomes a complicated function of t. This point will be reexamined in our discussion of the method of multiple time scales.

C. The Van der Pol oscillator

The next level of intricacy is attraction to or repulsion from a closed periodic orbit, a limit cycle. When the trajectory is attracted to an asymptotic closed periodic orbit, the latter is called a stable limit cycle. The Van der Pol oscillator is the usual system used to illustrate this type of behavior. If we use the complex variables, $z = x + iy$ and $z^* = x - iy$, Eq. (4) becomes

$$\dot{z} = -iz + \tfrac{1}{8}\epsilon\{4 - (z+z^*)^2\}(z - z^*), \tag{60}$$

where $\epsilon > 0$. For the discussion that follows, we present the z_n through $O(\epsilon^2)$ and U_n through $O(\epsilon^3)$,

$$z_1 = \tfrac{1}{4} i z_0^* - \tfrac{1}{16} i z_0^3 - \tfrac{1}{16} i z_0 z_0^{*2} - \tfrac{1}{32} i z_0^{*3} + f_1(z_0 z_0^*) z_0, \tag{61}$$

$$z_2 = -\tfrac{5}{1024} z_0^5 - \tfrac{1}{256} z_0^4 z_0^* + \tfrac{5}{512} z_0^2 z_0^{*3} + \tfrac{1}{1024} z_0 z_0^{*4} + \tfrac{5}{1536} z_0^{*5} - \tfrac{5}{128} z_0^3 - \tfrac{1}{32} z_0 z_0^{*2} + \tfrac{1}{128} z_0^{*3} + f_2(z_0 z_0^*) z_0, \tag{62}$$

$$U_1 = \tfrac{1}{2}(1 - \tfrac{1}{4} z_0 z_0^*) z_0, \tag{63}$$

$$U_2 = \tfrac{11}{256} i z_0^3 z_0^{*2} - \tfrac{3}{16} i z_0^2 z_0^* + \tfrac{1}{8} i z_0 + (-\tfrac{1}{4}\,\mathrm{Re}\, f_1 - f_1'(1 - \tfrac{1}{4} z_0 z_0^*)) z_0^2 z_0^* . \tag{64}$$

The minimal normal form option that U_2 vanishes identically requires that f_1 have a pole at $z_0 z_0^* = 4$, and therefore is not a good choice. Worse than that, a nonzero $\mathrm{Re}\, f_1$ in Eq. (64) introduces a spurious time dependence in the amplitude. The best choice for the free term in z_1 is $f_1 = 0$, which leads to

$$U_2 = \tfrac{11}{256} i z_0^3 z_0^{*2} - \tfrac{3}{16} i z_0^2 z_0^* + \tfrac{1}{8} i z_0, \tag{65}$$

and

$$U_3 = -\tfrac{13}{8192} z_0^4 z_0^{*3} + \tfrac{11}{1024} z_0^3 z_0^{*2} - \tfrac{3}{128} z_0^2 z_0^* + \{-\tfrac{1}{4}\,\mathrm{Re}\, f_2 - f_2'(1 - \tfrac{1}{4} z_0 z_0^*)\} z_0^2 z_0^* . \tag{66}$$

We first examine the simple choice of $f_2 = 0$. In polar coordinates, Eq. (32), the radial equation is

$$\frac{d\rho}{dt} = \frac{1}{2}\epsilon\rho\left(1 - \frac{1}{4}\rho^2\right) + \epsilon^3\left(-\frac{1}{8192}\rho^7 + \frac{11}{1024}\rho^5 - \frac{3}{128}\rho^3\right) + O(\epsilon^5). \tag{67}$$

Starting at small amplitudes, the system executes a spiral leading asymptotically to a stable limit cycle. In an $O(\epsilon)$ calculation, the right-hand side of Eq. (67) vanishes at $\rho^2 = 4$, defining the radius of the limit circle. However, in

higher orders, the zero of the right-hand side of Eq. (67) is shifted to

$$\rho^2 = 4 - \tfrac{3}{16}\epsilon^2 + O(\epsilon^4). \tag{68}$$

Thus, the radius of the limit circle is affected by the perturbation.

The equation for the (time-dependent) frequency is

$$\omega = \frac{d\theta}{dt} = 1 - \epsilon^2\left(\frac{11}{256}\rho^4 - \frac{3}{16}\rho^2 + \frac{1}{8}\right) + O(\epsilon^4). \tag{69}$$

To obtain the asymptotic value of the frequency as $t \to \infty$, we must substitute in Eq. (69) the asymptotic value of ρ, which is given by an infinite series in ϵ. This complication can be avoided by a radius renormalization choice for f_2. We look for an f_2 that will make the asymptotic value of ρ^2 remain at 4 when the third-order contribution is included in the amplitude equation. To this end, we need to make U_3 proportional to U_1. With U_3 given by Eq. (66), this behavior is achieved by

$$\mathrm{Re}\, f_2 = -\tfrac{13}{8192}(z_0 z_0^*)^3 + \tfrac{11}{1024}(z_0 z_0^*)^2 - \tfrac{3}{128}(z_0 z_0^*) \qquad (\mathrm{Im}\, f_2 = 0). \tag{70}$$

The expression for U_3 becomes

$$U_3 = (\tfrac{13}{4096} z_0^4 z_0^{*3} - \tfrac{11}{1024} z_0^3 z_0^{*2})(1 - \tfrac{1}{4} z_0 z_0^*). \tag{71}$$

The radial equation is modified and Eq. (67) is replaced by

$$\frac{d\rho}{dt} = \frac{1}{2}\epsilon\rho\left(1 - \frac{1}{4}\rho^2\right) \times \left\{1 + \epsilon^2\left(\frac{13}{2048}\rho^6 - \frac{11}{512}\rho^4\right) + O(\epsilon^4)\right\}. \tag{72}$$

Thus, the value of the limit-circle radius is unaffected by higher-order corrections and remains fixed at $\rho = 2$. As a result, the series for the asymptotic expression for the frequency is readily obtained:

$$\omega = 1 - \tfrac{1}{16}\epsilon^2 + O(\epsilon^4). \tag{73}$$

A detailed examination[17] shows that, as expected, shifting terms in the expansion procedure from the near-identity transformation to the normal form does not alter the physics. The free functions enable us to simplify parts of the calculation. However, all choices yield the same numerical accuracy through a given order.

In Fig. 4 a phase-plane plot of two solutions of Eq. (4) is shown: one spiraling toward the limit cycle from the outside, and one spiraling from the inside. To see the effect of the higher-order terms on the solution, a limit circle of $\rho = 2$ also is plotted.

Problem 4. Solve Eq. (4) for the same initial conditions as in Fig. 4, but with $\epsilon = -0.3$. What is the behavior of the solution?

V. THE METHOD OF MULTIPLE TIME SCALES

To motivate the introduction of the method of multiple time scales, we observe that in stable oscillatory systems, the time dependence of the solution is given through a phase, θ, which has the form

$$\theta = \omega t = \left(\sum_{n \geqslant 0} \epsilon^n \omega_n\right) t. \tag{74}$$

See, for example, Eq. (35) for the Duffing oscillator, and Eq. (73) for the $t \to \infty$ behavior of the van der Pol limit cycle. For $|\epsilon| \ll 1$ and short times $[t = O(1)]$, only the $n = 0$ term is important, because the terms $\epsilon^n t (n \geqslant 1)$ in Eq. (76) are negligibly small and hardly affect the time dependence. As $t \to O(1/\epsilon)$, the first two terms $(n = 0,1)$ become significant; when it becomes $O(1/\epsilon^2)$ the terms with $(n = 0,1,2)$ are significant, and so on. Thus, as time increases as measured in powers of $1/\epsilon$, a progressively larger number of terms in the time dependence become significant. The nth time scale is defined by

$$T_n = \epsilon^n t. \tag{75}$$

When Eq. (74) applies, the phase factor, $\exp(-i\theta)$, can be written in a factorized form:

$$e^{-i\theta} = e^{-i\omega_0 t} e^{-i\omega_1(\epsilon t)} e^{-i\omega_2(\epsilon^2 t)} \cdots = e^{-i\omega_0 T_0} e^{-i\omega_1 T_1} e^{-i\omega_2 T_2} \cdots. \tag{76}$$

For a given duration, hardly anything happens on sufficiently long time scales (for example, as the second hand of a clock moves a few seconds, the minute and hour hands hardly move). We are, therefore, tempted to treat the various time scales, T_n, as independent variables. This assumption is the starting point of the method of multiple time scales: The solution is assumed to depend on all time scales, $z = z(T_0, T_1, T_2, \ldots)$. Hence, in addition to expanding the solution in powers of ϵ [the near-identity transformation, Eq. (8)], the time derivative is replaced by:

$$\frac{d}{dt} \to \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots. \tag{77}$$

The introduction of independent time scales and the elimination of secular terms lead to conditions on the dependence of the solution on the various time scales (solvability conditions). For most dynamical systems, these conditions constitute nontrivial consistency constraints on the structure of the approximate solutions. In general, the freedom, enjoyed in the method of normal forms, to choose the free resonant terms that appear in every order in the near-identity transformation, Eq. (8), does not exist in the multiple time scales method. The consistency problems encountered in the method are an issue of much concern.[39–46]

For conciseness, it is customary to use the notation

$$D_n \equiv \frac{\partial}{\partial T_n}. \tag{78}$$

If we substitute Eqs. (8) and (77) in Eq. (7), and expand through $O(\epsilon^2)$, we obtain

$$(D_0 + \epsilon D_1 + \epsilon^2 D_2)(z_0 + \epsilon z_1 + \epsilon^2 z_2) = -i(z_0 + \epsilon z_1 + \epsilon^2 z_2) + i\epsilon F(z_0, z_0^*) + i\epsilon^2\left\{\frac{\partial F(z_0, z_0^*)}{\partial z_0} z_1 + \frac{\partial F(z_0, z_0^*)}{\partial z_0^*} z_1^*\right\}. \tag{79}$$

We collect terms order by order and obtain

$$D_0 z_0 + i z_0 = 0, \tag{80}$$

$$D_0 z_1 + i z_1 = -D_1 z_0 + F(z_0, z_0^*), \tag{81}$$

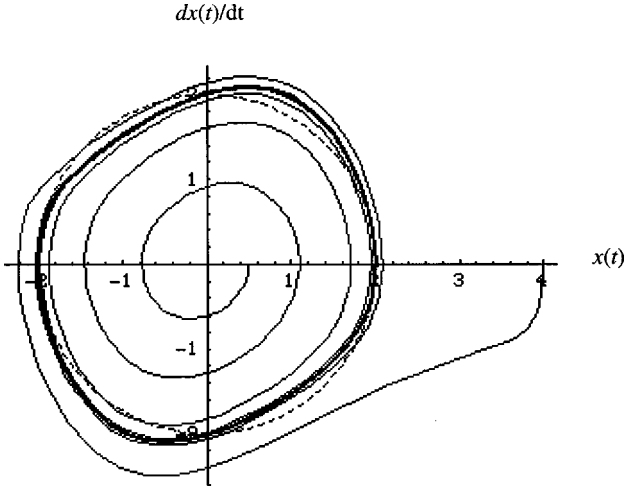

Fig. 4. Phase-plane plot of a solution of the Van der Pol oscillator with $\epsilon=0.3$. Solution of Eq. (4.35) (full line), limit cycle (thick lime), limit circle with radius=2 (dashed line).

$$D_0 z_2 + i z_2 = -D_1 z_1 - D_2 z_0 + \frac{\partial F(z_0, z_0^*)}{\partial z_0} z_1 + \frac{\partial F(z_0, z_0^*)}{\partial z_0^*} z_1^*. \tag{82}$$

Equation (80) is readily solved by

$$z_0 = A(T_1, T_2) e^{-iT_0}. \tag{83}$$

The amplitude, A, depends on all higher time scales. However, because we will limit our analysis to second order, only T_1 and T_2 have been written out explicitly.

Although the analysis of Eq. (81) depends on the detailed structure of the perturbation, $F(z_0, z_0^*)$, the following can already be said at this point. First, the homogeneous part of the equation,

$$D_0 z_1 + i z_1 = 0, \tag{84}$$

introduces a contribution z_1 which we write in the form

$$B(T_1) e^{-iT_0}. \tag{85}$$

The amplitude B may depend on all higher time scales. However, in a second-order analysis, only its T_1 dependence counts. Second, any term on the right-hand side of Eq. (81) whose T_0 dependence is of the form $\exp(-iT_0)$ resonates with the left-hand side of the equation, and hence will generate in z_1 a term that is linear in T_0 (a secular term). Such terms must be eliminated to yield an approximation that is valid for times of $O(1/\epsilon)$ and not just of $O(1)$. There is an explicit term of this kind, namely, $-D_1 z_0$. In addition, any monomial of the form $z_0^{n+1}\, z_0^{*n}$ in $F(z_0, z_0^*)$ will have this T_0 dependence. We denote the resonant part of $F(z_0, z_0^*)$ by $F(z_0, z_0^*)_R$. To eliminate all the terms that generate secular behavior, we impose the requirement

$$-D_1 z_0 + F(z_0, z_0^*)_R = 0. \tag{86}$$

This first-order solvability condition determines the T_1 dependence of the amplitude A of Eq. (83). Once the resonant term in Eq. (81) has been removed, we can find the full solution for z_1, including the solution of the homogeneous equation:

$$z_1 = \tilde{z}_1 + B(T_1) e^{-iT_0}, \tag{87}$$

where $\tilde{z}_1$ is the particular solution that accounts for the nonresonant contribution of $F(z_0, z_0^*)$ in Eq. (81).

With z_1 known, Eq. (82) is analyzed in a similar manner, and z_2 also will include a solution of the homogeneous part of Eq. (82). To avoid secular terms, the contribution of all the resonant terms on the right-hand side of Eq. (82) has to vanish, leading to the second-order solvability condition:

$$\left(-D_1 z_1 - D_2 z_0 + \frac{\partial F(z_0, z_0^*)}{\partial z_0} z_1 + \frac{\partial F(z_0, z_0^*)}{\partial z_0^*} z_1^* \right)_R = 0. \tag{88}$$

Equation (88) determines the T_1 dependence of the amplitude B. Consistency between Eqs. (86) and (88) will turn out to be a major limitation on the application of the method of multiple time scales.

As in the method of normal forms [see Eqs. (11) and (12)], a comment is in order regarding error estimates. An error of the magnitude defined by Eq. (11) is obtained if the near-identity transformation contains higher-order corrections through $O(\epsilon^N)$, as in Eq. (10). Concurrently, for each term, z_k, $k \leqslant N$, in the approximate solution, we find the dependence of z_k on T_0, $T_1, \ldots, T_{N+1-k}$. Thus, we have to find the dependence of z_0 on T_0, $T_1, \ldots, T_{N+1}$, of z_1 on T_0, $T_1, \ldots, T_N$, of z_2 on T_0, $T_1, \ldots, T_{N-1}$, etc. A major point that we will discuss in the following is that often the method of multiple time scales does not yield the dependence on the higher time scales, and we must augment the method with additional information to find that dependence.

VI. APPLICATIONS OF THE METHOD OF MULTIPLE TIME SCALES

A. The Duffing oscillator

For $F(z_0, z_0^*)$ of the Duffing oscillator [see Eq. (6)], and (83) for z_0, Eq. (81) becomes

$$D_0 z_1 + i z_1 = -D_1 A e^{-iT_0} - \tfrac{1}{8} i A^3 e^{-3iT_0} - \tfrac{3}{8} i A^2 A^* e^{-iT_0} - \tfrac{3}{8} i A A^{*2} e^{iT_0} - \tfrac{1}{8} i A^{*3} e^{3iT_0}. \tag{89}$$

The first and third terms on the right-hand side of Eq. (89) are the resonant terms. Their elimination yields the first-order solvability condition,

$$D_1 A = -\tfrac{3}{8} i A^2 A^*. \tag{90}$$

If we write A in the form,

$$A = |A| e^{-i\varphi_A}, \tag{91}$$

we readily find

$$D_1 |A| = 0, \quad \varphi_A = \tfrac{3}{8} |A|^2 T_1 + \psi_A(T_2). \tag{92}$$

Thus, the magnitude of the amplitude A is independent of T_1, and hence its phase is linear in T_1.

Now that we have eliminated the resonant terms in Eq. (89), the solution for z_1 is readily found to be

$$z_1 = \tfrac{1}{16} A^3 e^{-3iT_0} - \tfrac{3}{16} A A^{*2} e^{iT_0} - \tfrac{1}{32} A^{*3} e^{3iT_0} + B e^{-iT_0}. \tag{93}$$

We turn to the $O(\epsilon^2)$ equation, Eq. (82), and substitute Eqs. (90) and (93), together with $F(z_0,z_0^*)$ of Eq. (6), in Eq. (82), to obtain

$$\begin{aligned} D_0 z_2 + i z_2 = & -D_1\{\tfrac{1}{16}A^3 e^{-3iT_0} - \tfrac{3}{16}AA^{*2}e^{iT_0} \\ & - \tfrac{1}{32}A^{*3}e^{3iT_0} + Be^{-iT_0}\} - D_2 A e^{-iT_0} \\ & + \tfrac{3}{8}(Ae^{-iT_0} + A^* e^{iT_0})^2\{\tfrac{1}{32}A^3 e^{-3iT_0} \\ & - \tfrac{3}{16}A^2 A^* e^{-iT_0} - \tfrac{3}{16}AA^{*2}e^{iT_0} + \tfrac{1}{32}A^{*3}e^{3iT_0} \\ & + Be^{-iT_0}\}. \end{aligned} \tag{94}$$

To avoid secular terms in z_2, all the terms that are proportional to $\exp(-iT_0)$ must be eliminated from Eq. (94). This condition yields the second-order solvability condition:

$$D_2 A + D_1 B = \tfrac{51}{256} i A^3 A^{*2} - \tfrac{3}{4} i A A^* B - \tfrac{3}{8} i A^2 B^*. \tag{95}$$

Equation (95) and its complex conjugate may be regarded as linear, first-order differential equations for the T_1 dependence of B and B^*. However, because it contains the derivative of the amplitude A with respect to T_2, it also can be viewed as a constraint on A. Thus, we must check if the two solvability constraints, Eqs. (90) and (95), are mutually consistent. This check can be done by either solving Eqs. (90) and (95) directly or by requiring that derivatives of A with respect to T_2 and T_1 commute,

$$D_1 D_2 A = D_2 D_1 A, \tag{96}$$

when the derivative D_2 is applied to Eq. (90) and D_1 is applied to Eq. (95). For the Duffing oscillator, both ways are simple and easy to implement, while in more complicated systems, one way may be more advantageous than the other. We now perform the analysis in these two ways.

Problem 5. Derive Eq. (95).

Direct solution of solvability conditions. If we multiply Eq. (97) by A^* and its complex conjugate by A and use Eq. (90), we obtain

$$D_1(A^*B + AB^*) + D_2(AA^*) = 0, \tag{97}$$

$$\begin{aligned} & D_1(A^*B - AB^*) - 2iAA^* D_2 \varphi_A \\ & \quad = -\tfrac{3}{4} i A A^*(A^*B + AB^*) + \tfrac{51}{128} i (AA^*)^3. \end{aligned} \tag{98}$$

Because $|A|$ is independent of T_1 [see Eq. (92)], the term $D_2(AA^*)$ in Eq. (97) also is independent of T_1. Hence, the solution for $A^*B + AB^*$ is

$$(A^*B + AB^*) = -D_2(AA^*)T_1 + H(T_2, T_3, \ldots). \tag{99}$$

Consequently, Eq. (98) generates in $(A^*B - AB^*)$ terms that are linear and quadratic in T_1:

$$\begin{aligned} (A^*B - AB^*) = i\{ & (2AA^* D_2 \varphi_A + \tfrac{51}{128}(AA^*)^3 \\ & - \tfrac{3}{4} AA^* H(T_2, T_3, \ldots))T_1 \\ & + \tfrac{3}{8} AA^* D_2(AA^*)T_1^2 + J(T_2, T_3, \ldots)\}. \end{aligned} \tag{100}$$

Equations (99) and (100) yield the general solution of B with H and J real-valued functions.

Over the time span $t = O(1/\epsilon)$ for which the approximation is valid, powers of T_1 in B need not be eliminated: They do not spoil the ordering of the expansion in terms of progressively smaller terms, because ϵB contains contributions proportional to

$$\epsilon T_1^q = \epsilon(\epsilon t)^q \underset{t = O(1/\epsilon)}{\approx} O(\epsilon). \tag{101}$$

Problem 6. What happens to the magnitude of $(A^*B + AB^*)$ of Eq. (99) for $t = O(1/\epsilon^2)$?

The T_2 dependence of A is undetermined by Eqs. (90) and (95). To specify it, we must either provide additional input, extraneous to the method, for example, initial data for $T_1 = 0$. A common way to fix the T_2 dependence of A is to require that the solution be constructed solely out of periodic functions. To avoid aperiodic terms, Eq. (97) must be broken into

$$D_2(AA^*) = 0, \quad D_1(A^*B + AB^*) = 0, \tag{102}$$

which implies that

$$(A^*B + AB^*) = H(T_2, T_3, \ldots). \tag{103}$$

Hence, $|A|$ is independent also of T_2. If we use the same reasoning in Eq. (98), we must require

$$\begin{aligned} \varphi_A = & \frac{3}{8}(AA^*)T_1 - \frac{51}{256}(AA^*)^2 T_2 \\ & + \int_0^{T_2} \frac{3}{8}(A^*B + AB^*)dT_2 + \chi_A(T_3, T_4, \ldots), \end{aligned} \tag{104}$$

and

$$D_1(A^*B - AB^*) = 0, \tag{105}$$

which implies

$$(A^*B - AB^*) = iJ(T_2, T_3, \ldots). \tag{106}$$

Consequently, we find that, through $O(\epsilon^2)$, the dependence of A on T_1 and T_2 factors into a product of phase terms, each depending on one time scale only [although the T_2 dependence of $(A^*B + AB^*)$ is still unknown]:

$$\begin{aligned} A = |A| e^{-iT_0} \exp\left(-i\frac{3}{8}(AA^*)T_1\right) \exp\Bigg(i\Bigg\{ & \frac{51}{256}(AA^*)^2 T_2 \\ & - \int_0^{T_2} \frac{3}{8}(A^*B + AB^*)dT_2 \Bigg\}\Bigg) e^{-i\chi_A(T_3, T_4, \ldots)}. \end{aligned} \tag{107}$$

This pattern recurs in higher orders if we require that the solvability conditions be satisfied solely by periodic functions. If this requirement is not imposed, then the dependence of the amplitudes on higher time scales, T_n, $n \geqslant 2$, depends on our choice of the initial data at, say, $T_1 = 0$.

With the choice that the solution does not contain powers of T_1, Eqs. (103), (106), and (90) lead to

$$D_1(A^*B)=0$$
$$\Rightarrow D_1 B$$
$$=-\frac{3}{8}iAA^*B$$
$$\Rightarrow\begin{cases}B(T_1,T_2,\ldots)=|B|e^{-i(3/8)(AA^*)T_1}e^{i\psi_B(T_2,T_3,\ldots)}\\ D_1(BB^*)=0\end{cases} \tag{108}$$

Note that the T_2 dependence of A still cannot be determined unless additional input is provided. Part of the dependence has been found due to the requirement that, through $O(\epsilon^2)$, the solution does not contain powers of T_1. This requirement led to the result that $|A|$ does not depend on T_2. However, it is not obvious that the choices made up to this point are consistent with the solvability conditions in higher orders. Because we have limited our analysis to second order, we only state the answer: For the case of the Duffing oscillator (and all single-oscillator systems with energy conserving perturbations), the choices are consistent. Finally, we still do not know the dependence of φ_A, the phase of A, on T_2. Help comes from observing that, in this order, the solvability conditions leave $|B|$ free, allowing, in particular, $|B|=0$. This choice completes the T_2 dependence of A and yields a simple expression for φ_A:

$$\varphi_A=-\tfrac{3}{8}(AA^*)T_1+\tfrac{51}{256}(AA^*)^2T_2+\chi_A(T_3,T_4,\ldots). \tag{109}$$

Note that the T_1 dependence of φ_B, the phase of B, is the same as that of φ_A, leading to another choice for B that has aesthetic appeal (and also is consistent with the higher-order solvability conditions):

$$B=\tfrac{17}{64}A^2A^*. \tag{110}$$

With this choice. φ_A, and, hence, A, become independent of T_2, and Eq. (104) becomes

$$\varphi_A=\tfrac{3}{8}(AA^*)T_1+\chi_A(T_3,T_4,\ldots). \tag{111}$$

In the analysis in higher orders, the requirement that the dependence of the amplitudes be expressed in terms of only periodic functions (that is, exponential phase factors), implies that A, B, and the free amplitudes that appear in higher orders will all have the same time dependent phases.

Consistency of solvability conditions. The constraint for consistency between the two solvability conditions is obtained by applying D_2 to Eq. (90), D_1 to Eq. (95), and requiring that Eq. (96) be obeyed [using Eq. (95) repeatedly]. The result yields the T_1 dependence of B:

$$D_1^2B+\tfrac{3}{2}i(AA^*)D_1B+\tfrac{3}{4}iA^2D_1B^*-\tfrac{27}{64}(AA^*)^2B$$
$$+\tfrac{9}{32}(AA^*)A^2B^*=0. \tag{112}$$

Because Eq. (112) is homogeneous, $B=0$ is an allowed choice. Similarly, the use of Eq. (90) shows that the choice of Eq. (110) also is an allowed solution of Eq. (112).

We end our analysis of the Duffing oscillator by noting that the multiple time scales results are completely parallel to those of the method of normal forms. For instance, in the latter method, the free term in z_1 is a resonant term, that is, proportional to z_0 [see Eq. (26)]. The equivalent statement in the multiple time scales method, if the time dependence of the solution is constructed only from periodic functions (that is, there are no powers of T_1 in the solution), is that B has the same phase as A. Also, the choice $B=0$ is the equivalent to the choice $\alpha=0$ in Eq. (26), and the choice of Eq. (110) is the equivalent of the minimal normal form choice of $\alpha=17/64$, mentioned toward the end of Sec. IV A.

B. The oscillator with cubic damping

The Duffing oscillator is representative of dynamical systems to which an energy conserving perturbation is added. The methods of normal forms and multiple time scales are completely equivalent, and there is a one-to-one correspondence of the results. In particular, any choice of the free terms in the normal form analysis has a corresponding choice in the multiple time scales analysis and both methods yield the same functional form for the approximate solution. The two examples of dissipative systems to be discussed in the following represent obstacles encountered in the method of multiple time scales, which are not encountered in the normal form analysis. These difficulties occur in the multiple time scales analysis of most linear dynamical systems that are subject to a nonlinear perturbation, be they systems of one degree of freedom with a nonconservative perturbation, or Hamiltonian systems with more than one degree of freedom.

Because the analysis has been presented in detail for the Duffing oscillator, we skip some of the steps in the analysis of the cubically damped oscillator, and only discuss the results and the important points. The zero-order solution, z_0, is given by Eq. (83). The first-order correction is found to be

$$z_1=\tfrac{1}{16}iA^3e^{-3iT_0}-\tfrac{3}{16}iAA^{*2}e^{iT_0}+\tfrac{1}{32}iA^{*3}e^{3iT_0}+Be^{-iT_0}. \tag{113}$$

The first- and second-order solvability conditions are

$$D_1A=-(\tfrac{3}{8})A^2A^*, \tag{114}$$

$$D_2A+D_1B=\tfrac{27}{256}iA^3A^{*2}-\tfrac{3}{4}AA^*B-\tfrac{3}{8}A^2B^*. \tag{115}$$

Direct solution of solvability conditions. If we write A as in Eq. (91), Eq. (114) is solved by

$$D_1(AA^*)=-\tfrac{3}{4}(AA^*)^2,\quad D_1\varphi_A=0, \tag{116}$$

leading to

$$A=\sqrt{(AA^*)_0\Big/\left(1+\frac{3}{4}(AA^*)_0T_1\right)}e^{-i\varphi_A(T_2,T_3)}, \tag{117}$$

where $(AA^*)_0$ does not depend on T_1. If we multiply Eq. (115) by A^*, and its complex conjugate equation by A, and sum or subtract the results, we find after repeated use of Eqs. (114) and (115):

$$D_1\{(A^*B+AB^*)/(AA^*)^2\}-D_2(1/(AA^*))=0, \tag{118}$$

$$D_1\{(A^*B-AB^*)/(AA^*)+\tfrac{9}{32}i(AA^*)\}+2iD_2\varphi_A=0. \tag{119}$$

We use Eq. (117) and solve Eq. (118) to find

$$(A^*B+AB^*)=\{D_2(1/(AA^*)_0)T_1+H(T_2,\ldots)\}$$
$$\times(AA^*)^2. \tag{120}$$

The T_1 linear term in Eq. (120) does not spoil the ordering of the magnitude of the zero- and first-order terms, because

thanks to Eq. (117), this term generates $O(T_1^{-1})$ leading behavior in (A^*B+AB^*).

Equation (119) is solved by

$$(A^*B-AB^*)=\{-\tfrac{9}{32}i(AA^*)-2iD_2\varphi_A T_1 + iJ(T_2,\dots)\}(AA^*). \tag{121}$$

By using Eqs. (12) and (121), the solution for B is found to be

$$B=\{[\tfrac{1}{2}H(T_2,\dots)-\tfrac{9}{64}i]AA^*+\tfrac{1}{2}[D_2(1/(AA^*)_0AA^*T_1 + iJ(T_2,\dots))]\}A-iD_2\varphi_A AT_1. \tag{122}$$

Here, H and J are real valued functions of higher time scales. Again, the T_2 dependence of A is not specified; its determination requires additional input that is not provided by the method of multiple time scales and may be fixed as follows. The last term in Eq. (122) generates a term proportional to $T_1^{1/2}$ in B. For the allowed time span, $t=O(1/\epsilon)$, for which the approximation is valid, this term does not destroy the ordering in the near-identity transformation. However, because a $T_1^{1/2}$ term does not agree with the expected structure of the approximate solution, it makes sense to eliminate it by requiring that

$$D_2\varphi_A=0. \tag{123}$$

With this choice, there is no secular behavior in B, because its asymptotic T_1 dependence becomes that of A, namely, $O(T_1^{-1/2})$. Hence, there is no constraint on $D_2(1/(AA^*)_0)$ in Eq. (122), and, without additional input, we cannot determine the T_2 dependence of A. An appealing choice is to assume that A does not depend on T_2. To this end, we have to add to Eq. (123) the requirement that

$$D_2(1/(AA^*)_0)=0. \tag{124}$$

Equation (122) for B then becomes

$$B=\{\tfrac{1}{2}H(T_2,\dots)-\tfrac{9}{64}i\}A^2A^*+\tfrac{1}{2}iJ(T_2,\dots)A. \tag{125}$$

Note that the first-order minimal normal form choice of the normal form analysis, Eq. (55), is a particular case of Eq. (125), with $J=H=0$. This latter choice is consistent with the two solvability conditions. It turns out that the analysis in higher orders allows for the cancellation of the dependence of A on the higher time scales $(T_3,\dots)$ as well. This cancellation is a reflection of the fact that for the cubically damped oscillator, only two time scales, T_0 and T_1, are required to describe the physics. However, the equivalence between the methods of normal forms and multiple time scales ends at this point. For example, in the normal form analysis we could chose the free resonant term to vanish, but here $B=0$ is allowed only in the trivial case, $A=0$, as can be deduced directly by a detailed inspection of Eqs. (114) and (115), or the solution for B given by Eq. (122).

Problem 7. What happens to the magnitude of B of Eq. (122) for $t=O(1/\epsilon^2)$? Use Eq. (117).

Consistency of solvability conditions. The same conclusions regarding the constraints on B may be reached more easily by the requirement that the two solvability conditions, Eqs. (114) and (115), are mutually consistent. If we apply D_2 to Eq. (114) and D_1 to Eq. (115) and impose Eq. (96), we obtain:

$$D_1^2B+\tfrac{3}{2}(AA^*)D_1B+\tfrac{3}{4}A^2D_1B^*+\tfrac{9}{64}(AA^*)^2B + \tfrac{9}{32}(AA^*)A^2B^*+\tfrac{81}{512}iA^4A^{*3}=0. \tag{126}$$

Clearly, $B=0$ is allowed only for the trivial case $A=0$. On the other hand, a detailed check shows that the choice of Eq. (125) is an allowed solution of Eq. (126), and hence is consistent with both solvability conditions, Eqs. (114) and (115).

In summary, we have to overcome two stumbling blocks. First, unlike the Duffing oscillator, the freedom of choice of the free amplitudes, which the normal form analysis enjoys, is lost due to the requirement that the solvability conditions for different orders be mutually consistent. Second, as for the Duffing oscillator, the method of multiple time scales does not provide the dependence of the solution on T_2, or, for that matter, any time scales beyond T_1, without additional information, consistent with the solvability conditions.

Problem 8. Prove the statement following Eq. (126).

C. The Van der Pol oscillator

We follow the steps given in detail in Sec. IV, employ the same notation, and obtain the first-order results:

$$D_1A=\tfrac{1}{2}A(1-\tfrac{1}{4}AA^*), \tag{127}$$

$$z_1=-\tfrac{1}{16}iA^3e^{-3iT_0}+\{-\tfrac{1}{16}AA^{*2}+\tfrac{1}{4}A^*\}ie^{iT_0} - \tfrac{1}{32}iA^{*3}e^{3iT_0}+Be^{-iT_0}. \tag{128}$$

The second-order analysis yields

$$D_1B+D_2A=\tfrac{1}{2}B-\tfrac{1}{4}AA^*B-\tfrac{1}{8}A^2B^*+\tfrac{1}{8}iA-\tfrac{3}{16}iA^2A^* + \tfrac{11}{256}iA^3A^{*2}, \tag{129}$$

$$z_2=-\tfrac{5}{1024}A^5e^{-5iT_0}+\{\tfrac{1}{256}A^4A^*-\tfrac{5}{128}A^3-\tfrac{3}{16}iA^2B\}e^{-3iT_0} +\{\tfrac{5}{512}A^2A^{*3}-\tfrac{1}{32}AA^{*2}+\tfrac{1}{4}iB^*-\tfrac{1}{16}iA^{*2}B -\tfrac{1}{8}iAA^*B^*\}e^{iT_0}+\{\tfrac{1}{128}A^{*3}+\tfrac{1}{1024}AA^{*4} -\tfrac{3}{32}iA^{*2}B^*\}e^{3iT_0}+\tfrac{5}{1536}A^{*5}e^{5iT_0}+Ce^{-iT_0}. \tag{130}$$

Direct solution of solvability conditions. Equation (127) yields

$$D_1(AA^*)=AA^*\left(1-\frac{1}{4}AA^*\right), \tag{131}$$

$$D_1\varphi_A=0. \tag{132}$$

Equation (131) is solved by

$$AA^*=\frac{4(AA^*)_0}{((AA^*)_0+(4-(AA^*)_0)\exp(-T_1))}. \tag{133}$$

φ_A and $(AA^*)_0$ do not depend on T_1, but may be functions of T_2, $T_3,\dots$.

We multiply Eq. (129) by A^* and employ Eqs. (127), (129), (132), and (133), and obtain for the real and imaginary parts:

$$D_1\left\{\frac{\exp(T_1)(A^*B+AB^*)}{(AA^*)^2}\right\} =D_2\left(\frac{\exp(T_1)}{(AA^*)}\right)=D_2\left(\frac{1}{(AA^*)_0}\right), \tag{134}$$

$$D_2\varphi_A-\frac{1}{16}=D_1\Bigg\{-\frac{(A^*B-AB^*)}{2iAA^*}+\frac{1}{16}\log(AA^*)-\frac{11}{64}(AA^*)\Bigg\}. \tag{135}$$

Equation (134) is solved by

$$(A^*B+AB^*)=\exp(-T_1)\Bigg(G(T_2,T_3,\dots)+D_2\Bigg(\frac{1}{(AA^*)_0}\Bigg)T_1\Bigg)(AA^*)^2, \tag{136}$$

where G is a free real-valued function. Although all the terms in Eq. (136) fall off exponentially, their ordering in the perturbation series is destroyed if the T_1 linear term is not eliminated in Eq. (136). To eliminate this term, we impose

$$D_2(AA^*)_0=0\Rightarrow D_2(AA^*)=0, \tag{137}$$

$$(A^*B+AB^*)=\exp(-T_1)G(T_2,T_3,\dots)(AA^*)^2. \tag{138}$$

We note that the left-hand side of Eq. (135) is independent of T_1 [see Eq. (132)]. The integration of Eq. (135) over T_1, therefore, yields (with H a real valued function):

$$\Bigg(D_2\varphi_A-\frac{1}{16}\Bigg)T_1=-\frac{(A^*B-A\overline{B})}{2iAA^*}+\tfrac{1}{16}\log(AA^*)-\tfrac{11}{64}(AA^*)+H(T_2,T_3,\dots). \tag{139}$$

The last three terms on the right-hand side of Eq. (139) cannot counterbalance the linear behavior of the left-hand side for $T_1\to\infty$. Hence, to avoid a T_1 secular term in B, the left-hand side of Eq. (139) must vanish, yielding

$$D_2\varphi_A=\tfrac{1}{16}\Rightarrow\varphi_A=\tfrac{1}{16}T_2+\psi_A(T_3,T_4,\dots), \tag{140}$$

$$(A^*B-AB^*)=\{\tfrac{1}{16}\log(AA^*)-\tfrac{11}{64}(AA^*)+H(T_2,T_3,\dots)\}2i(AA^*). \tag{141}$$

From Eqs. (138) and (141), we find

$$B=\{\tfrac{1}{2}G(T_2,T_3,\dots)\exp(-T_1)(AA^*)+iH(T_2,T_3,\dots)+\tfrac{1}{16}i\log(AA^*)-\tfrac{11}{64}i(AA^*)\}A. \tag{142}$$

The characteristics of the solution are obviously the same as in the method of normal forms. Namely, through second order, the zero-order term asymptotes to a limit circle of radius 2. Had we continued the analysis to third order, we would have found an ϵ^2 correction to the radius of the limit circle. However, here comes the difference. In the method of normal forms, we saw that it is possible to choose the free term in first order to vanish [f_1 of Eq. (61)]. The equivalent requirement here is

$$B\equiv 0. \tag{143}$$

However, unlike the energy conserving Duffing oscillator, the solvability conditions impose restrictions on this choice. From Eq. (142) we find that due to the T_1 dependence of A, Eq. (143) is satisfied only if the solution vanishes identically or is already on the limit cycle:

$$A=0, \tag{144}$$

$$AA^*\equiv 4. \tag{145}$$

The choice of Eq. (145) makes AA^* independent of all higher time scales. That is, the zero-order term must be chosen as the asymptotic circle with radius 2:

$$A=2e^{i((1/16)T_2+\psi_A(T_3,T_4,\dots))}. \tag{146}$$

Equations (145) and (146) imply that z_0 and z_1 are both confined to their asymptotic limit. Hence, this choice does not cover situations in which the initial condition is removed from the limit cycle by $O(\epsilon)$ or $O(1)$, so that exponential spiraling toward the limit cycle occurs only in orders higher than first.

The T_2 dependence of the phase, φ_A, given by Eq. (140), is required for consistency with the solvability conditions. In the normal forms method, this behavior is attained only as $T_2\to\infty$. For example, if we use Eqs. (63) and (64) for U_1 and U_2, respectively, the normal form, Eq. (9) becomes

$$\dot{z}_0=-iz_0+\tfrac{1}{2}\epsilon(1-\tfrac{1}{4}z_0z_0^*)z_0+\epsilon^2\{-[\tfrac{1}{4}\mathrm{Re}\,f_1(z_0z_0^*)+f_1'(z_0z_0^*)(1-\tfrac{1}{4}z_0z_0^*)]z_0^2z_0^*+\tfrac{11}{256}iz_0^3z_0^{*2}-\tfrac{3}{16}iz_0^2z_0^*+\tfrac{1}{8}iz_0\}. \tag{147}$$

The first-order term in Eq. (147) ensures that $\rho\to 2$ as $t\to\infty$. Through second order, the time dependence of the phase is

$$\frac{d\varphi_A}{dt}=-\epsilon^2\Bigg\{\frac{11}{256}\rho^4-\frac{3}{16}\rho^2+\frac{1}{8}-\mathrm{Im}\,f_1'(\rho^2)\Bigg(1-\frac{1}{4}\rho^2\Bigg)\rho^2\Bigg\}. \tag{148}$$

Thus, in general, Eq. (140) of the method of multiple time scales is reached only for $t\to\infty$. To obtain Eq. (140) as the exact ϵ^2 dependence of φ_A in the normal forms analysis, we must choose for the free function, f_1:

$$\mathrm{Im}\,f_1(\rho^2)=\tfrac{1}{16}\ln\rho^2-\tfrac{11}{64}\rho^2+C, \tag{149}$$

where C is an arbitrary constant. In the normal forms method this is a possible choice. Equation (142) shows that in multiple time scales it is the only possibility.

Consistency of solvability conditions. If we apply the derivative D_1 to Eq. (129) and D_2 to Eq. (127) and use both equations repeatedly, we find that Eq. (96) leads to:

$$D_1^2B-D_1B+\tfrac{1}{2}AA^*D_1B+\tfrac{1}{4}A^2D_1B^*+\tfrac{1}{4}B+\tfrac{1}{64}(AA^*)^2B+\tfrac{1}{32}A^3A^*B^*+\tfrac{3}{16}iA^2A^*-\tfrac{17}{128}iA^3A^{*2}+\tfrac{11}{512}iA^4A^{*3}=0. \tag{150}$$

If we multiply Eq. (150) by A^*, we obtain for the real and imaginary parts:

$$D_1^2\{\exp(T_1)(A^*B+AB^*)/(AA^*)^2\}=0, \tag{151}$$

$$D_1^2\Bigg\{\frac{(A^*B-AB^*)}{(AA^*)}+\frac{1}{8}i\log(AA^*)-\frac{11}{32}iAA^*\Bigg\}=0. \tag{152}$$

A detailed analysis shows that Eqs. (151) and (152) are consistent with Eqs. (134) and (139), respectively, from which the solution for B, Eq. (142), has been obtained. Hence, the consistency test is equivalent to the direct analysis of the solution of the solvability conditions. In particular, the choice $B=0$, is consistent with Eq. (152) only for $A=0$ or $AA^*=4$.

VII. CONCLUDING COMMENTS

Our goal has been to give a tutorial on the application of two commonly used perturbation methods, normal forms and multiple time scales. To simplify the discussion, the analysis has been confined to single oscillator systems that are subject to nonlinear perturbations. We have shown the effects of typical nonlinear perturbations: modification of the period for energy conserving perturbations and damping- or limit-cycle phenomena for dissipative perturbations. Both perturbation methods focus on the manner in which the perturbation modifies the zero-order approximation in comparison with the naive expansion. In both methods, the effect of the perturbation on the solution for the zero-order approximation is identified as the resonant terms that are generated by the perturbation in every order. If, as is done in the naive expansion, they are not accounted for by their effect on the zero-order term, they lead to the emergence of secular terms. The goal is to generate a zero-order approximation that contains the major physical effects of the perturbation, but is still a small modification of the solution of the unperturbed equation.

Our analysis has stressed the role played by the freedom inherent in perturbation theory. There is a one-to-one correspondence between the exploitation of this freedom in the application of the two methods to a harmonic oscillator that is modified by an energy conserving perturbation. This correspondence is lost when the methods are applied to dissipative systems for which only a subset of the choices available in the method of normal forms is allowed in the method of multiple time scales.

Because both methods have been applied to a much wider class of systems, we know that the loss of freedom in the multiple time scales method extends beyond the systems with a single degree of freedom and applies to most dynamical systems of physical interest. We provide here an intuitive argument. Consider a system of two harmonic oscillators that have equal unperturbed frequencies and are coupled by an energy conserving interaction. The combined system is energy conserving. However, because of the equality of the unperturbed frequencies, energy flows efficiently back and forth between the two oscillators due to the coupling between them. ''Efficiently'' means that, although the coupling is weak, characterized by a small parameter, ϵ, the energy flow between the two oscillators, while slow [becoming significant only for $t=O(1/\epsilon)$], is not small. Thus, each subsystem looks like a dissipative one. The multiple time scales analysis of this system reveals that the freedom of choice of the free amplitudes is limited by the requirement that the solvability conditions for different orders be mutually consistent.[46] The same applies to the application of the multiple time scales method to systems described by PDEs, because they are systems with an infinite number of degrees of freedom. (For example, expanding the solution in a Fourier series in the spatial coordinate, leads to an infinite set of ODEs for the time dependence of the Fourier coefficients.) The lesson from the examples discussed here is that in every application of the method that goes beyond a first-order analysis, the choices made for the perturbative solution must ensure that the solvability conditions for different orders be obeyed in a mutually consistent manner.

The application of the normal forms method to systems of physical interest of spatial extent encounters other unexpected difficulties. For example, if a small nonlinear perturbation is added to the Burgers equation (which describes the propagation of shock waves in a fluid or traffic flow on a highway) or the Korteweg–de Vries equation (which describes the propagation of shallow water waves), then new consistency problems called ''obstacles to integrability'' are encountered.[18–22] The unperturbed equations have interesting nonlinear wave solutions (for example, solitons). Often, the perturbative expansion generates terms that spoil the simple structure of the unperturbed solution, and the zero-order approximation (which is determined by the normal form) may not have the same wave structure. The resolution of this open problem is under investigation.[47]

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[1]H. Poincaré, *New Methods of Celestial Mechanics*, originally published as *Les Methodes Nouvelles de la Mechanique Celeste* (Gauthier-Villars, Paris, 1892) and (AIP Press, New York, 1993).
[2]G. D. Birkhoff, *Dynamical Systems* (American Mathematical Society Colloquium, New York, 1927).
[3]G. I. Hori, ''Theory of general perturbations with unspecified canonical variables,'' Publ. Astron. Soc. Jpn. **18**, 287–296 (1966).
[4]A. Deprit, ''Canonical transformations depending on a parameter,'' Celest. Mech. **1**, 12–13 (1969).
[5]G. I. Hori, ''Theory of general perturbations for non-canonical systems,'' Publ. Astron. Soc. Jpn. **23**, 567–587 (1971).
[6]V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1988), 2nd ed.
[7]A. D. Bruno, *Local Methods in Nonlinear Differential Equations* (Springer-Verlag, New York, 1989).
[8]A. H. Nayfeh, *Method of Normal Forms* (Wiley, New York, 1993).
[9]J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1988).
[10]S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer-Verlag, New York, 1990).
[11]A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).
[12]C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw–Hill, New York, 1978).
[13]A. H. Nayfeh, *Introduction to Perturbation Techniques* (Wiley, New York, 1981).
[14]J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer-Verlag, New York, 1985).
[15]J. A. Murdock, *Perturbation Theory and Methods* (Wiley, New York, 1991).
[16]J. Kevorkian and J. D. Cole, *Multiple Scale and Singular Perturbation Methods* (Springer-Verlag, New York, 1996).
[17]P. B. Kahn and Y. Zarmi, *Nonlinear-Dynamics: Exploration Through Normal Forms* (Wiley, New York, 1998).
[18]Y. Kodama and T. Taniuti, ''Higher order approximation in the reductive perturbative method,'' J. Phys. Soc. Jpn. **45**, 298–310 (1978).
[19]A. S. Fokas and Q. M. Liu, ''Asymptotic integrability of water waves,'' Phys. Rev. Lett. **77**, 2347–2351 (1996).
[20]Y. Kodama, ''Normal form and solitons,'' in *Topics in Soliton Theory and Exactly Solvable Nonlinear Equation*, edited by M. J. Ablowitz *et al.* (World Scientific, Singapore, 1987), pp. 319–340.
[21]R. A. Kraenkel, J. G. Pereira, and E. C. de Rey Neto, ''Linearizability of the perturbed Burgers equation,'' Phys. Rev. E **58**, 2526–2530 (1998).
[22]Y. Hiraoka and Y. Kodama, ''Normal form and solitons,'' Lecture notes, Euro Summer School 2001, The Isaac Newton Institute, Cambridge, 15–25 August, 2002 and arXiv:nlin.SI/0206021.
[23]A. C. Newell, ''Envelope equations,'' Lect. Appl. Math. **15**, 157–163 (1974).
[24]Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, New York, 1984).
[25]P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic, New York, 1990).
[26]C. M. Cross and P. C. Hohenberg, ''Pattern formation outside of equilibrium,'' Rev. Mod. Phys. **65**, 851–112 (1993).
[27]B. Van der Pol, ''On oscillation hysteresis in a simple triode generator,'' Philos. Mag. **43**, 700–719 (1926).

[28]Lord Rayleigh (J. W. Strutt), ''On maintained vibrations,'' Philos. Mag. **XV**, 229–232 (1883).
[29]M. Kummer, ''How to avoid secular terms in classical and quantum mechanics,'' Nuovo Cimento Soc. Ital. Fis., B **1B**, 123–129 (1971).
[30]J. C. Liu, ''The uniqueness of normal forms via Lie transformations and its application to Hamiltonian systems,'' Celest. Mech. **36**, 89–95 (1985).
[31]J. C. Van der Meer, *The Hamiltonian Hopf Bifurcation* (Springer-Verlag, New York, 1985).
[32]A. Baider and R. C. Churchill, ''Uniqueness and non-uniqueness of non-uniqueness of normal form expansions for vector fields,'' Proc. R. Soc. Edinburgh, Sect. A: Math. **108**, 27–33 (1988).
[33]A. Baider, ''Unique normal forms for vector fields and Hamiltonians,'' J. Diff. Eqns. **78**, 33–39 (1989).
[34]P. B. Kahn and Y. Zarmi, ''Minimal normal forms in harmonic oscillations with small nonlinear perturbations,'' Physica D **54**, 65–74 (1991).
[35]P. B. Kahn and Y. Zarmi, ''Radius renormalization in limit cycles,'' Proc. R. Soc. London, Ser. A **440**, 189–199 (1993).
[36]P. B. Kahn, D. Murray, and Y. Zarmi, ''Freedom in small parameter expansion for nonlinear perturbations,'' Proc. R. Soc. London, Ser. A **443**, 83–94 (1993).
[37]E. Forest and D. Murray, ''Freedom in minimal normal forms,'' Physica D **74**, 181–196 (1994).
[38]P. B. Kahn and Y. Zarmi, ''Reduction of secular error in approximations to harmonic oscillator systems with nonlinear perturbations,'' Physica D **118**, 221–249 (1998).
[39]Z. Rahman and T. Burton, ''On higher order methods of multiple scales in non-linear oscillators—Periodic steady state response,'' J. Sound Vib. **133**, 369–379 (1989).
[40]J. Murdock, ''Some fundamental issues in multiple scale theory,'' Appl. Anal. **53**, 157–173 (1994).
[41]Y. Kodama, ''Normal forms, symmetry and infinite-dimensional Lie algebra for systems of ODE's,'' Phys. Lett. A **191**, 223–228 (1994).
[42]J. Murdock and L. C. Wang, ''Validity of the multiple scale method for very long intervals,'' ZAMP **47**, 760–789 (1996).
[43]A. Degasperis, S. V. Manakov, and S. M. Santini, ''Multiple-scale perturbation beyond the nonlinear Schrödinger equation. I,'' Physica D **100**, 187–211 (1997).
[44]C. Rega, W. Lacarbonara, A. H. Nayfeh, and C. M. Chin, ''Multiple resonance in suspended cables: Direct versus reduced-order models,'' Int. J. Non-Linear Mech. **34**, 901–924 (1999).
[45]G. Auberson and M. Capdequi Peyranère, ''Quantum anharmonic oscillator in the Heisenberg picture and multiple scale techniques,'' Phys. Rev. A **65**, 032120-1–9 (2002).
[46]P. B. Kahn and Y. Zarmi (unpublished).
[47]A. Veksler and Y. Zarmi (unpublished).