

1. Circular motion and centripetal force

Newton's First Law told us that an object continues to move in a straight line unless acted upon by an external force. A special case of an external force arises in the case of an object that moves, at a constant rate, along a circular path. In order for this to happen, it is necessary to apply a force of constant magnitude that acts at all times perpendicular to the direction in which the object is moving at that moment. **Figure 1** illustrates the situation at three successive instants of time, t_1 , t_2 and t_3 .

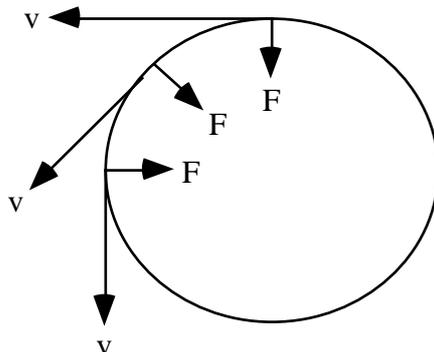


Figure 1: A particle, under the action of a force of constant magnitude that is directed always towards the same point, moves with a velocity of constant magnitude but continuously changing direction, in a circle around that point.

Notice that the magnitudes of both the velocity v and the force F remain constant: Only their directions change with time. Velocity and force are two examples of so-called **vector** quantities, i.e. quantities that have magnitude and direction. This is in contrast to quantities, like mass and time, which have magnitude but no direction. Those kinds of quantities are called **scalar** quantities.

Now we could, in principle, perform an experiment to measure the magnitude of the force F that is responsible for circular motion. We could attach a spring balance (of the kind that measures the *weight*, i.e. forces) to a string of known length r . At the other end of the string we would attach a weight of known mass m . We would then twirl the mass around our head, in a circular path and at a constant rate which we would measure with a clock. A friend would then take a photograph that would enable us to read the force indicated by the spring balance. We could perform many such experiments; some with different weights on the end of the string; some with strings of different lengths; and some at different rates of revolution. When we come to analyse the results we would find that a simple formula relates the mass, string length and rate of rotation to the magnitude of the force. It is:

$$F = m v^2 / r \quad (4.1)$$

You will notice that this formula is of the type: “force = mass x something”, where the “something” is v^2/r . But from Newton's 2nd Law, this must be the acceleration of the mass.

“How strange!” you are probably thinking. The mass moves with a velocity of constant magnitude and yet it is accelerating. But this is not strange. We defined acceleration as the rate of change of the velocity, and in this case the velocity is changing - in direction! If the force had been acting from behind, it would increase the magnitude of the object’s velocity (i.e. speed it up). Conversely, if the force had been acting from ahead, it would decrease the magnitude of the object’s velocity (i.e. slow it down). If the force had acted from behind at some arbitrary angle to the object’s motion, it would both speed it up and change its direction. Similarly, if the force had been acting from ahead but at some arbitrary direction to the object’s line of motion, it would both slow it down and change its direction. Clearly then, if the force acts precisely perpendicularly to the line of the object’s motion, it would neither slow it down nor speed it up: It would *only* change its direction. This is the case we have for circular motion. For such motion, the acceleration is:

$$a = v^2 / r \tag{4.2}$$

Just like velocity, acceleration is also a vector quantity. In the present situation the magnitude of the acceleration remains constant but its direction changes continuously. However, at all moments, the direction of the acceleration a is in the same direction as the force F that causes it.

The force F , in eq. (4.1) is sometimes called a *centripetal* force from the Greek for “causing to move in a circle”.

2. Centripetal forces and gravitation

Let us now take a look at how knowledge of the centripetal force enables us to make all kinds of interesting astronomical calculations. As our first example, consider a planet of mass m , moving in a circular orbit of radius r about the sun of mass M . From Newton’s law of gravity we know that the centripetal force (i.e. the force of gravity that maintains the planet in orbit around the sun) takes the value:

$$F = GMm/r^2 \tag{4.3}$$

where G is Newton’s gravitational constant, which we encountered in the previous lecture.

By equating eq. (4.1) and eq. (4.3) we see that the mass of the planet cancels out and we are left with a relationship between the radius of the planet’s orbit and the magnitude of its velocity:

$$r v^2 = GM \tag{4.4}$$

Now the velocity in orbit is usually not a much-used parameter. Instead, one usually talks about the *period* T of a planet. This is the time it takes to complete one orbit (i.e. the length of a year on that planet). Clearly, the relationship between T and v is:

$$v = 2\pi r / T \tag{4.5}$$

If we insert eq. (4.5) into eq. (4.4) we obtain the relation:

$$r^3 / T^2 = GM / 4\pi^2 \tag{4.6}$$

Eq. (4.6) is remarkable in that it applies equally to *any* planet, no matter what its mass m happens to be.

Let us check it for the earth. From astronomical tables, the earth-sun mean radius is $r = 1.496 \times 10^{11}$ m, and the average length of 1 year is $T = 3.156 \times 10^7$ s. Therefore the left-hand side of eq. (4.6) = $3.361 \times 10^{18} \text{ m}^3 \text{ s}^{-2}$. On the other hand, $G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, the sun's mass is $M = 1.989 \times 10^{30}$ kg. Therefore, the right-hand side of eq. (4.6) = $3.362 \times 10^{18} \text{ m}^3 \text{ s}^{-2}$. The small difference is due to the fact that I have rounded all of the data to 3 decimal places.

Furthermore, eq. (4.6) does not only apply to our sun and its planets. If M is the mass of some distant star, eq. (4.6) enables astronomers to calculate the orbital radius of one of its planets if the period is known. Often, the planet is far too close to its parent sun to enable it be resolved by a telescope but one observes its periodicity via changes in brightness of the light emitted by the star.

Eq. (4.6) also applies to the motion of the moon around the earth and, indeed, also to that of any artificial earth satellite.

3. Fictitious Forces

Suppose we were to hold an empty picture frame in our hands and view the world through it. Everything would appear pretty normal, unless we started to jerk the frame in arbitrary directions. In that situation, the world would appear to be jerking about too - like a movie made by a bad cameraman. If we didn't realize that the frame was jerking and were to try and understand the strange motion of the objects we were looking at, we might arrive at the erroneous conclusion that some kind of weird forces were at work - perhaps an earthquake or something. These forces would, of course, be **fictitious** - purely the result of viewing the world from an accelerating frame of reference.

In the above example I have exaggerated in order to make a point. If, instead of a jerking picture frame, we were to view the earth from the window of an accelerating rocket ship, we could compute the earth's relative motion to us by assuming that a fictitious force was accelerating the earth away from us. We would know that the force *is* fictitious because accelerometers on board our ship (and also our bodies) would tell us that *we* were accelerating, whereas accelerometers on the earth would record nothing unusual down there. Remember that velocity is relative but acceleration is absolute. In order for there to be no fictitious forces we must use a so-called **inertial** reference frame, i.e. one which is not accelerating.

Now let us return to the earth. Because the earth rotates about its axis, it is actually an accelerating reference frame. Fortunately, for most purposes, the resulting fictitious forces are very small. If this were not the case, Newton might never have discovered his famous laws! But there are situations in which the fictitious forces associated with the earth's rotation do become manifest. Two such fictitious forces are of interest to geographers. One is the so-called **centrifugal force**, which appears to act even on stationary objects. The other is **coriolis force**, which becomes apparent when objects move.

3.1 Centrifugal force

This is the easier of the two fictitious forces to understand, because we experience it in everyday life. Each time we round a leftward bend in an automobile we “feel” a fictitious centrifugal force trying to throw us sideways to the right. In fact, what we actually feel is the reactive force of the car wall pushing us to the left when, by Newton’s First Law, we are trying to continue in a straight line. Quantitatively, centrifugal force is numerically equal to centripetal force but in the opposite direction.

In our previous discussion of planetary motion around the sun, instead of saying that the centripetal force is the gravitational force, as we did when we derived eq. (4.4), we could have obtained the same result by using the following reasoning: *When a planet revolves around the sun, there are two forces acting on it which must balance each other precisely. One is the force of gravity, pulling it towards the sun. The other is the centrifugal force, throwing it away from the sun.* It doesn’t matter which kind of reasoning you prefer to use as they both give the same answers.

Let me give two examples in which the concept of centrifugal force helps provide a simple intuitive explanation for two geographical phenomena. The first, is the shape of the earth, which is not perfectly spherical: While the young earth was cooling down but still molten, its rotational motion about its own axis created a centrifugal force that tried to eject material into space (mostly towards the equator, where the magnitude of the rotational velocity is greatest). The resulting equatorial radius (6377 km) is thus slightly larger than the polar radius (6356 km). This difference in radii is one factor that causes the precise value of g , the acceleration due to gravity, to vary slightly at different places on earth: At the poles, we are nearer to the center of the earth than we would be at the equator.

A second factor that causes g to take a slightly different value at the equator may also easily be understood by thinking in terms of centrifugal force: When we stand at the equator, gravity pulls us down with an acceleration of, say, 9.81 m s^{-2} ; but the centrifugal force caused by the earth’s rotation tries to throw us out into space with an acceleration of v^2/R , where R is the radius of the earth and v is the tangential velocity of the earth’s surface as it rotates on its axis once every 24 hours.

$$v = 2\pi \times 6.377 \times 10^6 \text{ m} / (24 \times 60 \times 60 \text{ s}) = 463.7 \text{ m s}^{-1}$$

Hence,

$$v^2/R = (463.7 \text{ m s}^{-1})^2 / (6.377 \times 10^6 \text{ m}) = 0.03 \text{ m s}^{-2}.$$

We see thus, that the earth’s rotational motion *reduces* the value of g by 0.03 m s^{-2} , compared to what it would be at the poles.

3.2 Coriolis force

When you sit in a children’s merry-go-round it is easy to “feel” the centrifugal force, fictitious as it is. But if you try walking in a straight you will be in for another surprise. In order to see what happens; let us examine the case of a marble being rolled across a rotating turntable. We shall assume that there is no friction between the marble and the turntable, so that the marble does, indeed, travel in a straight line, as demanded by Newton’s First Law,

and as would be evident to an observer looking down on the marble from above. **Figure 2** shows three successive instants of time, t_1 , t_2 and t_3 , starting from the projection of the marble from point A towards point B.

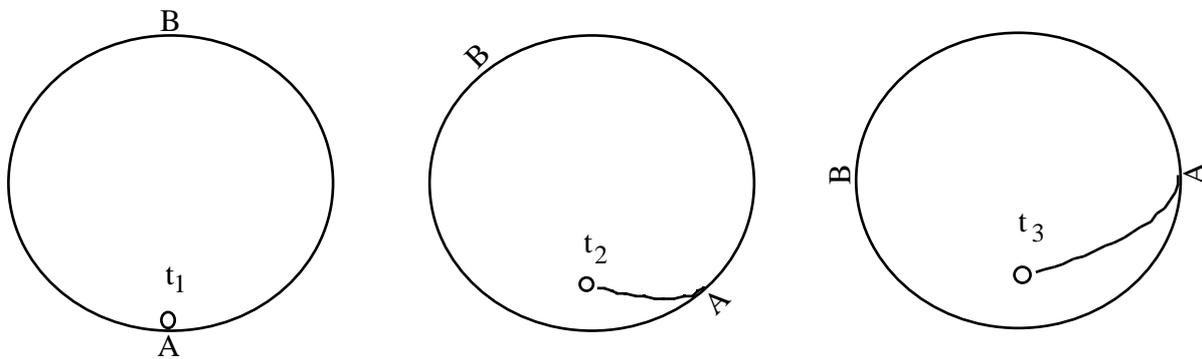


Figure 2: The marble moves in a straight line, at constant velocity, along the direction that AB pointed at time t_1 . In real space (as seen by us, who look down upon the turntable) the marble continues along this straight path, see at subsequent instants t_2 and t_3 . However, to someone turning with the turntable, the path of the marble appears to be a complicated curve, as shown in the figure. The rotating observer interprets this curved path as being due to a fictitious coriolis force.

In **Figure 2** I have also drawn a curve which indicates the path that the marble would appear to follow if viewed by an observer who was rotating with the turntable. Such an observer would interpret the non-linear path as being due to a fictitious force, which is called the coriolis force.

Unlike the centrifugal force, which is easy to describe quantitatively, the mathematical description of the coriolis force is too complicated to derive in this course. So I shall not write it down. The coriolis force due to the earth's rotation does have a number of physical manifestations in situations where trajectories are very long (e.g. rivers that flow great distances in a north-south direction, or wind regimes), or for situations in which velocities are very high (e.g. for long range military projectiles). However, it is too weak a force to manifest itself on the small scale (e.g. for bathwater running down the drain!).

Problem set 4 (forces and gravity):

1. Communication satellites move around the earth, above the equator, at the same rate at which the earth rotates on its axis. In this way the satellite appears to stand stationary at a fixed point in the sky. Such an orbit is called a **geostationary orbit**. Calculate the height above the earth's surface, in km, of a geostationary satellite. [For this calculation you should take: $G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, the mass of the earth $M = 5.976 \times 10^{24} \text{ kg}$, the mean radius of the earth $R = 6371 \text{ km}$, and one day = 24 hours].
2. Look up the orbital parameters r and T for all of the planets in the solar system and examine the extent to which eq.(4.6) is valid for each of them.
3. The river Nile flows approximately from south to north, from near the equator to a latitude of approximately 30° N . On which bank would you expect to find more erosion, the left or the right? Explain why.