Spontaneously Induced General Relativity:

The Electromagnetic Case

Thesis submitted in partial fulfillment of the requirement for the degree of master of science in the Faculty of Natural Sciences

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Abstract

Spontaneous symmetry breaking (SSB) has played a crucial role in physics, from condensed matter physics (super conductivity) all the way to elementary particle physics (electro/nuclear interactions). Having in mind that even general relativity (GR) is a spontaneously induced theory, rather than the fundamental theory of gravity, the extension of the SSB mechanism to gravity naturally suggests that the reciprocal Newton's constant is the vacuum expectation value (VEV) of some scalar field.

It has been recently demonstrated by Davidson and Gurwich that spontaneously induced GR does not necessarily admit the full general relativistic limit. In particular, it has been shown that the black hole limit is governed by a phase transition which occurs precisely at the would have been event horizon. Whereas the general relativistic exterior Schwarzschild solution is fully recovered, it connects now, by means of a smooth self similar transition profile, with a novel interior core exhibiting a variety of new features.

In this thesis, we extend the Davidson-Gurwich scheme by the inclusion the electromagnetic interaction. As expected, the recovered exterior general relativistic Reissner Nordström geometry connects now with a non general relativistic core. The latter is serendipitously characterized by: (i) Absence of a signature flip, (ii) Locally varying effective Newton's constant, (iii) Vanishing spatial volume, (iv) Constant surface gravity, (v) Closing light cone structure, (vi) Rindler structure near the origin, (vii) Jordan/Einstein frame independence, and in some respects, resembles a maximally stretched horizon. It turns out that the Komar mass residing inside any concentric interior sphere is proportional to the surface area of that sphere, and consequently, is non-negative definite and furthermore non-singular at the origin. This is accompanied by the exact Hawking Euclidean time periodicity (with the conic defect unconventionally defused at the origin rather at the would have been horizon). Combining these two ingredients, we show that associated with any inner sphere of circumferential radius r is the total purely geometrical universal entropy $S(r) = \pi r^2/G$. The corresponding holographic entropy packing locally saturates the 't Hooft-Susskind-Bousso universal holographic bound.

Acknowledgements

Thank the guy who said $c = \hbar = k_B = 1$.

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Contents

1 Introduction

It has been recently demonstrated by Davidson and Gurwich that spontaneously induced GR does not necessarily admit the full general relativistic limit [1]. In particular, it has been shown that the black hole limit is governed by a phase transition which occurs precisely at the would have been event horizon. Whereas the general relativistic exterior Schwarzschild solution is fully recovered, it connects now, by means of a smooth self similar transition profile, with a novel interior core exhibiting a variety of new features[2]. In this thesis, we extend the Davidson-Gurwich scheme by the inclusion the electromagnetic interaction. The main results can be found in [3].

The theoretical background of this issue is rather basic and most of it can be found in graduate level text books. I found [4] [5] [6] [7] extremely helpful. Black hole thermodynamics go little behind the the standard text book material, but comprehensive reviews are at hand [8] [9] [10].

1.1 ϕR Gravity

ϕR theories of gravity are modifications, or rather generalizations, of Einsteins General Relativity (GR). It was usually most recognized with Brans-Dickie scalar field theory from the early 60's [11], and gained renewed interest in the 90's due to some developments in string theories [12] [13]. An important class of gravitational theories which is equivalent to ϕR gravity is the $f(R)$ gravity [14].

In simple words, the basic idea is that rather than taking Newton's constant as constant, we will give him a role of a dynamical scalar field. The scalar field is then coupled to Ricci scalar in the Lagrangian

$$\frac{1}{G}R \to \phi R \ . \tag{1.1}$$

We can add a potential $V(\phi)$ to the theory so that the scalar field can acquire VEV of $\phi = G^{-1}$, and then the theory is equivalent to GR. Another viable option is adding a kinetic term proportional to some ω [11]. The general relativistic limit is then obtain for

$\omega \to \infty$, which can be interpreted as a weak coupling to the scalar field, or making the field nondynamical.

Variation for a Lagrangian of that sort will lead us to a "modified Einstein equation"

$$\phi G_{\mu\nu} = 8\pi T^{eff}_{\mu\nu} \ . \tag{1.2}$$

It is possible to define a conformal transformation

$$\tilde{g}_{\mu\nu} = \phi g_{\mu\nu} \ , \tag{1.3}$$

so that Einstein equation will regain its canonical form

$$\tilde{G}_{\mu\nu} = 8\pi \tilde{T}^{eff}_{\mu\nu} \ . \tag{1.4}$$

The two different frames $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are known respectively as the Jordan frame and Einstein frame. There is a long standing debate as to which conformal frame is physical one [15] [16], and some even suggest that both of them apply [17]. Luckily enough, our main results, though derived (naturally) in the Jordan frame, will hold as well in Einstein's frame.

1.2 Reissner Nordström

The "canonical" example for a charged black hole in GR is the Reissner Nordström (RN) black hole. To be more precise, the RN metric is a static spherical-symmetric solution to Einstein's equations corresponding to the action

$$S = -\int d^4x\sqrt{-g}\left(\frac{R}{16\pi} + \frac{1}{4}F^2\right) \ . \tag{1.5}$$

The resulting metric is given by

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{1.6}$$

The RN metric holds one singularity for $r = 0$ as could be seen by checking Kretschmann scalar

$$\mathcal{K} = \frac{1}{r^4} \ , \tag{1.7}$$

which clearly diverges for $r = 0$.

The horizon is somewhat more complicated. The possible horizon should be defined by the equation

$$1 - \frac{2GM}{h} + \frac{GQ^2}{h^2} = 0 \ , \tag{1.8}$$

and we can mark $r_\pm = h_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$ as the solution. Now we have three distinct cases:

(i) $GM^2 < Q^2$: No horizon at all, a naked singularity emerges. r coordinate is always space-like.

(ii) $GM^2 = Q^2$: Extreme Reissner Nordström. There is a horizon, but the r coordinate is never time-like.

iii) $GM^2 > Q^2$: Two horizons. r coordinate is space-like between h_+ and h_- and time-like for $r < h_-$.

We will restrict our interest to case (iii). The metric coefficients of the metric are plotted in figure 1.

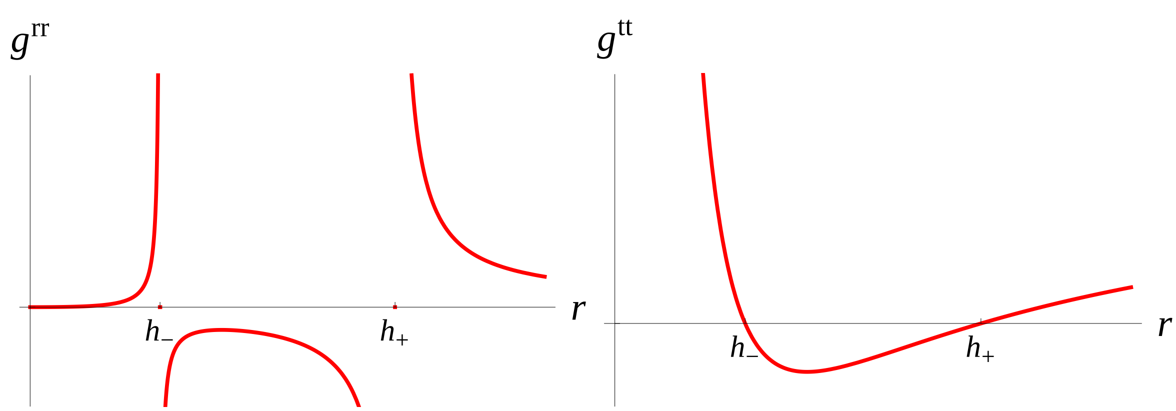

Figure 1: RN coefficients plots for RN solution with $GM^2 > Q^2$. The solution admits an outer horizon h_+ and inner one h_-. both of the coefficients change sign between h_+ and h_-, hence the $t \leftrightarrow r$ flip.

Both horizons suffer $t \leftrightarrow r$ flip. In the appropriate coordinate system (Eddington), we can see that the light cones tilt over at h_+, and between h_+ and h_- all future directed paths are in the direction of decreasing r, as in ordinary black hole. Below h_- light cones tilt back, and both directions are allowed. Thus, a observer entering the black hole by crossing the outer horizon, will inevitably fall below the lower horizon and than he can decide whether to continue toward the singularity in the origin or to cross again h_-. Choosing the latter, the observer re-enters the inter-regions. This time it will be a white hole, all future directed paths are in the direction the outer horizon, and the inter region "spit out" the observer. The conformal diagram of the RN metric is given in figure 2.

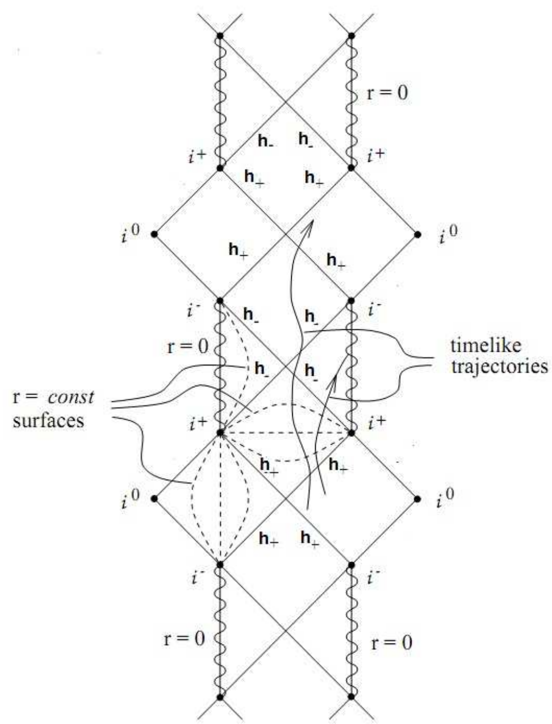

Figure 2: Conformal diagram for RN solution with $GM^2 > Q^2$.

1.3 Mass in General Relativity

General relativity does not offer a unique definition for mass, but offers several different definitions which are applicable under different circumstances. Since we are dealing with an asymptomatically flat space-time, we can reduce the number of relevant candidates.

The ADM energy is given by [18]

$$\frac{1}{16\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\sigma^i\left(\partial_j h_i^j - \partial_i h_j^j\right) \ , \tag{1.9}$$

where n^μ is the normal vector associated with hypersurface Σ, σ is the outward pointing normal vector of $\partial\Sigma$ and $\gamma^{(2)}$ is the induced metric on $\partial\Sigma$ which is taken to be at infinity, and so is irrelevant for measuring mass in some finite volume.

To define energy (mass) in a more local fashion, we can integrate the energy current J^μ over a spacelike surface Σ,

$$E = \int_\Sigma d^3x\sqrt{\gamma^{(3)}}n_\mu J^\mu \ . \tag{1.10}$$

The only question to ask is, what will be the current J^μ? A naive guess will be to use the energy momentum tensor: $J^\mu = K_\nu T^{\mu\nu}$, here K_ν is a time-like killing vector. There are several problems with that definition. For example, Schwarzschild $T^{\mu\nu}$ vanishes, so with that definition, Schwarzschild black hole will have zero mass. A different way to define the current is due to Komar [19], implementing Ricci tensor:

$$J^\mu = K_\nu R^{\mu\nu} \ . \tag{1.11}$$

With that choice of current, we can rewrite eq. (1.10) as

$$E = m_k = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla\mu K^\nu \ . \tag{1.12}$$

The above formula is known as Komar integral or Komar mass. To convince ourself that m_k is indeed a reasonable expression for the mass, we can calculate it for Schwarzschild black hole:

$$m_k^S = \frac{1}{4\pi G}\int d\theta d\phi r^2\sin\theta\frac{GM}{r^2} = M \ , \tag{1.13}$$

as could be expected. Note that for the RN metric, the result is

$$m_k^{RN} = M - \frac{Q}{r^2} \ . \tag{1.14}$$

For a stationary spherical symmetric metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \ . \tag{1.15}$$

one can simplify the result to

$$m_k = \frac{1}{2G} e^{\frac{\nu(r)-\lambda(r)}{2}} \nu'(r) r^2 \tag{1.16}$$

which will be useful in our future calculations.

1.4 Thermodynamics of Black Holes

Hawking area theorem [20] states that, under certain restrictions, the area of the event horizon is non-decreasing. Bekenstein pointed out [21] that this is analogues to the second law of thermodynamics. The entropy of the black hole is then proportional to the surface area

$$S = \frac{A}{4G} \ , \tag{1.17}$$

and the area theorem become the second law of *black hole thermodynamics*.

The first law of black hole thermodynamics for a rotating charged black hole is [10]

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ \ , \tag{1.18}$$

where κ is the surface gravity, Ω is the angular velocity and Φ is the electric potential. This relates between the change in area to the change in mass, angular momentum and charges for an adiabatic transition between nearby stationary black hole solutions. Neglecting for a moment the angular momentum and charge, we can compare this to the first law of thermodynamics $dE = TdS$. Taken with the second law of thermodynamics, we found the temperature as

$$T = \frac{\kappa}{2\pi} \ . \tag{1.19}$$

The temperature equation can rise naturally from totally different reason. Hawking discovered [22] that due to particle creation process on the horizon, the black hole emits radiation of a black body in temperature (1.19). We can explain this phenomena with the help of the Unruh effect, which states that an observer with constant acceleration a in Minkowski vacuum (Rindler Observer) observes thermal radiation of particles with

$$T = \frac{a}{2\pi} \ . \tag{1.20}$$

In the vicinity of the horizon, it is possible to recover Rindler structure for the metric. For example, near the horizon, at $r = h_+ + \delta r$, the RN metric is well approximated by

$$ds^2 \approx -\frac{2G}{h^2}\left(1 - \frac{GQ^2}{h^2}\right) dt^2 + \frac{h^2}{2G}\left(1 - \frac{GQ^2}{h^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{1.21}$$

Introducing the proper length by

$$d\eta = \frac{h^2}{2G}\left(1 - \frac{GQ^2}{h^2}\right)^{-1/2} dr \ , \tag{1.22}$$

we recover the Rindler structure next to the horizon

$$ds_{in}^2 = -\kappa^2 \eta^2 dt^2 + d\eta^2 + h^2 d\Omega^2 \ . \tag{1.23}$$

It is reasonable to assume, that the vacuum will be defined according to a free falling observer. A static observer at distance r_1 near the horizon will detect Unruh temperature (1.20) T_1. A second static observer located far from the black hole, at r_2 will see the temperature redshifted to

$$T_2 = \frac{V_1}{V_2} T_1 \ , \tag{1.24}$$

where $V(r)$ is the redshift factor

$$V(r) = \sqrt{1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}} \ . \tag{1.25}$$

If we take the second observer to infinity than $V_2 = 1$, and the temperature observed very far from the black hole is

$$T_\infty = \lim_{r_1 \to 2GM} \frac{V_1 a_1}{2\pi} = \frac{\kappa}{2\pi} \ , \tag{1.26}$$

as required.

2 Einstein Equation for Spontaneous Induced Gravity

The most simple action (in Jordan frame) that will lead us to spontaneous broken gravity with electromagnetism, must have the following ingridients: (i) Brans-Dickie scalar field coupled to Ricci scalar, (ii) Potential that will give the scalar field a VEV of the reciprocal of G and, of course, (iii) The electromagnetic tensor. Thus, the basic action for our theory is given by

$$S = -\int d^4x\sqrt{-g}\left[\frac{\phi R}{16\pi} + \frac{3}{32\pi a}\left(\phi - \frac{1}{G}\right)^2 + \frac{1}{4}F^2\right] \ . \tag{2.1}$$

One might wonder about the peculiar factor of the potential, or the absence of the kinetic term. We will discuss these issues later in section (2.2). For now, however, I would like to dive in and see what we can get from it.

2.1 Field Equations

In equation(2.1) we have three dynamic fields: ϕ, A_μ and $g^{\mu\nu}$. Variation according to the first two is trivial and respectively leads us to two equations of motion:

$$R + \frac{3}{a}\left(\phi - \frac{1}{G}\right) = 0 \ , \tag{2.2}$$

$$\nabla_\nu F^{\mu\nu} = 0 \ . \tag{2.3}$$

Variation with respect to the metric is much more complicated, and the exact calculation could be found in appendix B. The final result however gives us a ”modified Einstein equation”

$$\phi G_{\mu\nu} = -\phi_{;\mu\nu} + g_{\mu\nu}\Box\phi + \frac{3}{4a}\left(\phi - \frac{1}{G}\right)^2 g_{\mu\nu} - 2F_\mu{}^\alpha F_{\nu\alpha} + \frac{1}{2}F^2 g_{\mu\nu} \ , \tag{2.4}$$

The equations of motion are hard to handle and we can simplify them more by hypothesising a static spherical symmetric metric:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \ . \tag{2.5}$$

The electromagnetic tensor will have only two non-vanishing components:

$$F_{tr} = -F_{rt} = E(r) \ . \tag{2.6}$$

Starting with A_μ equation of motion (2.3), we find that under our static spherical assumptions it can be recasted into the form

$$E'(r) + \left(\frac{2}{r} - \frac{\nu'(r) + \lambda'(r)}{2}\right) E(r) = 0 \ . \tag{2.7}$$

Upon integration we obtain an expression for the electric field

$$E(r) = \frac{\tilde{Q}}{r^2} e^{\frac{\lambda(r)+\nu(r)}{2}} \ , \tag{2.8}$$

where $\tilde{Q}$ is some constant of integration, yet to be determine.

The next step will be to take the trace of equation(2.4). Multiplying it by $g^{\mu\nu}$

$$\begin{aligned}
\phi(R - 2R) &= -\Box\phi + 4\Box\phi + \frac{3}{a}\left(\phi - \frac{1}{G}\right)^2 - 2F_\mu{}^\alpha F^\mu{}_\alpha + 2F^2 \ , \\
\Box\phi &= \frac{1}{3}\left(-\phi R - \frac{3}{a}\left(\phi - \frac{1}{G}\right)^2\right) \ ,
\end{aligned}$$

and substituting eq. (2.2) gives us a Klein Gordon (KG) equation for ϕ

$$\Box\phi = \frac{1}{aG}\left(\phi - \frac{1}{G}\right) \ . \tag{2.9}$$

Using the KG equation (2.9) and the metric (2.5) in (2.17) will give us four equations, one for each diagonal term of the metric. We can manipulate the equations to get three coupled equations of $\phi(r)$, $\lambda(r)$ and $\nu(r)$ (see appendix C for details). The resulting equations are

$$\nu'(r) \quad = \frac{2\phi''(r)}{\left(\frac{2}{r}\phi(r) + \phi'(r)\right)} - \lambda'(r) \ , \tag{2.10}$$

$$\phi''(r) \quad + \left(\frac{2}{r} + \frac{\nu'(r) - \lambda'(r)}{2}\right)\phi'(r) - \frac{e^{\lambda(r)}}{aG}\left(\phi(r) - \frac{1}{G}\right) = 0 \ , \tag{2.11}$$

$$\begin{aligned}
\phi''(r) \quad &+ \frac{1}{2}\left(\lambda'(r) - \nu'[r]\right)\left(\frac{2}{r}\phi(r) - \phi'(r)\right) - \frac{2}{r^2}\left(1 - e^{\lambda(r)}\right)\phi(r) \ , \\
&- \frac{3e^{\lambda(r)}}{2a}\left(\phi(r) - \frac{1}{G}\right)\left(\phi(r) + \frac{1}{3G}\right) - 2\frac{e^{\lambda(r)}\tilde{Q}^2}{r^4} = 0 \ .
\end{aligned} \tag{2.12}$$

At this point, I will set $\tilde{Q}$ by demanding that for the limit of $\phi \to G^{-1}$ the metric coefficients e^{λ} and e^{ν} will take the form of the RN metric (1.6). So, the previous equations turn to be

$$\nu'(r) = -\lambda'(r) \ , \tag{2.13}$$

$$\frac{2\lambda'(r)}{rG} - \frac{2}{r^2 G}\left(1 - e^{\lambda(r)}\right) - 2e^{\lambda(r)}\frac{\tilde{Q}^2}{r^4} = 0 \ . \tag{2.14}$$

If we use the known expressions for RN

$$e^{\lambda(r)} = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} \ , \tag{2.15}$$

we find that

$$\tilde{Q}^2 = Q^2 \ , \tag{2.16}$$

which identifies Q as the charge.

2.2 Some Notes on the Lagrangian

2.2.1 The Potential

Let us re-derive the KG equation, this time for a general potential $V(\phi)$. In order to do so, we first write eq. (2.4) with a general potential:

$$\phi G_{\mu\nu} = -\phi_{;\mu\nu} + g_{\mu\nu}\Box\phi + \frac{1}{2}V(\phi)g_{\mu\nu} - 2F_{\mu}{}^{\alpha}F_{\nu\alpha} + \frac{1}{2}F^2 g_{\mu\nu} \ . \tag{2.17}$$

Tracing as before yield

$$\Box\phi = \frac{1}{3}\left(-\phi R - 2V(\phi)\right) \ . \tag{2.18}$$

Eq. (2.2) for a general potential is

$$R + \frac{dV}{d\phi} = 0 \ , \tag{2.19}$$

and together with the KG equation they form

$$\Box\phi = \frac{1}{3}\left(\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right) \ . \tag{2.20}$$

This is simply a Klein Gordon equation for a massless particle ϕ :

$$\Box\phi = \frac{1}{2}\frac{dV_{eff}}{d\phi} \ , \tag{2.21}$$

where the V_{eff} is given by

$$\frac{dV_{eff}}{d\phi} = \frac{2}{3}\left(\phi\frac{dV}{d\phi} - 2V(\phi)\right) \ . \tag{2.22}$$

For our choice of potential we get

$$V_{eff}(\phi) = \frac{1}{2aG}\left(\phi - \frac{1}{G}\right)^2 + const \ . \tag{2.23}$$

The similarity between the two potentials $V(\phi)$ and $V_{eff}(\phi)$ is not generic.

2.2.2 Kinetic Term

On simplicity ground, our basic action does not include a kinetic term for ϕ. This however, does not make the scalar field non-dynamical, as can be seen from KG equation. The reason for that is the coupling to Ricci scalar. Yet, I would like to show, that adding such kinetic term, will not change the dynamics too much.

A kinetic term for the Lagrangian will be (remember that the units of ϕ is m^{-2})

$$\mathcal{L}_k = \omega\frac{1}{4}g^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi} \ , \tag{2.24}$$

with some dimensionless ω to label the term contribution. Eq. (2.4) will gain extra terms:

$$\phi G_{\mu\nu} = \frac{\omega}{\phi}\left(\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\phi_{'\alpha}\phi_{'\beta} - \phi_{'\mu}\phi_{'\nu}\right) - \phi_{;\mu\nu} + ... \tag{2.25}$$

and tracing for the KG equation

$$\Box\phi = \frac{1}{2+3\omega}\frac{dV_{eff}}{d\phi} \ . \tag{2.26}$$

so this is just rescaling the potential (for $\omega \neq -3/2$). Remember that we have introduced in our potential a factor a for that exact reason.

3 Black Hole Emerging: Solution to the Field Equations

The aim of this section is to solve the equations of motions derived in the previous section. We will begin by summarizing the results:

$$\begin{aligned}
E(r) &= \frac{Q}{r^2} e^{\frac{\lambda(r)+\nu(r)}{2}} \ , && (3.1)\\
\nu'(r) &= \frac{2\phi''(r)}{\frac{2}{r}\phi(r)+\phi'(r)} - \lambda'(r) \ , && (3.2)\\
\phi''(r) &+ \left(\frac{2}{r} + \frac{\nu'(r)-\lambda'(r)}{2}\right)\phi'(r) - \frac{e^{\lambda(r)}}{aG}\left(\phi(r)-\frac{1}{G}\right) = 0 \ , && (3.3)\\
\phi''(r) &+ \frac{1}{2}\left(\lambda'(r)-\nu'[r]\right)\left(\frac{2}{r}\phi(r)-\phi'(r)\right) - \frac{2}{r^2}\left(1-e^{\lambda(r)}\right)\phi(r) \\
&- \frac{3e^{\lambda(r)}}{2a}\left(\phi(r)-\frac{1}{G}\right)\left(\phi(r)+\frac{1}{3G}\right) - \frac{2Q^2 e^{\lambda(r)}}{r^4} = 0 \ . && (3.4)
\end{aligned}$$

As we have seen is section (2.1), for $\phi = G^{-1}$ the solution is just the RN solution

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \ . \qquad (3.5)$$

We hereby tag this solution with some $\epsilon = 0$. In that language, this thesis is mainly devoted for the $\epsilon \to 0$ solution. One may expect that the RN solution will be recovered, but as we will see, this is not necessarily the case.

The above equations are rather complicated to solve, and just shoving them to Mathematica won't do the trick. Instead, we will solve eqs. (3.1)-(3.4) in a few steps. First, we solve them in two regions: very large r (section 3.1) and very small (section 3.2). Then, using the approximated solutions as boundary conditions, we will carry a numerical evaluation of the solution. This information will be enough for us to match the two solutions and find the solution structure on the phase transition (section 3.3).

3.1 Asymptotic Solution: Perturbated Reissner Nordström

Very far from the origin, we would expect that our solution will recover the RN solution (1.6). Consider thus a solution of the form of RN with small perturbation $s \to 0$:

$$\phi(r) = \frac{1}{G}\left(1 + s\phi_1(r)\right) \ , \tag{3.6}$$

$$\lambda(r) = -\mathrm{Log}\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + sL_1(r)\right) \ , \tag{3.7}$$

$$\nu(r) = \mathrm{Log}\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + sN_1(r)\right) \ . \tag{3.8}$$

Substituting into the field equations and expanding for small parameter s to first order yield

$$\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)\phi_1''(r) + \frac{2}{r}\left(1 - \frac{GM}{r}\right)\phi_1'(r) - \frac{1}{aG}\phi_1(r) = 0 \ , \tag{3.9}$$

$$L_1(r) + rL_1'(r) = \left(\frac{GQ^2}{r^2} - \frac{r^2}{aG}\right)\phi_1(r) + \frac{G}{r}\left(-Q^2 + Mr\right)\phi_1'(r) \ . \tag{3.10}$$

Note that the first equation is a decoupled equation for ϕ_1, and we will solve it first by using the variation method. For leading order in $1/r$, the equation will take the form

$$(\phi_1)_0''(r) + \frac{2}{r}(\phi_1)_0'(r) - \frac{1}{aG}(\phi_1)_0(r) = 0 \ , \tag{3.11}$$

with the solution given by

$$(\phi_1)_0 = C_1\frac{e^{-\sqrt{\frac{1}{aG}}r}}{r} + C_2\frac{e^{\sqrt{\frac{1}{aG}}r}}{r} \ . \tag{3.12}$$

We choose only the converging Yokawa tail part and set $C_2 = 0$. The solution of (3.9) will be variation of $(\phi_1)_0$

$$\phi_1(r) = \frac{e^{-\sqrt{\frac{1}{aG}}r}}{r}f(r) \ , \tag{3.13}$$

with $f(r)$ obeying the equation

$$\sqrt{aG}f''(r) - 2f'(r) - \frac{2\sqrt{aG}M}{ar}f(r) = 0 \ . \tag{3.14}$$

The solution is rather complicated, and given by terms Hyper-geometric and MeijerG functions. but asymptoticly can be written as (with an appropriate choice of the constants) as

$$f(r) = r^{-\sqrt{G/a}M} \ , \tag{3.15}$$

so that our final (asymptotic) result for ϕ_1 is

$$\phi_1(r) = \frac{e^{-\sqrt{\frac{1}{aG}}r}}{r^{1+\frac{GM}{\sqrt{aG}}}} \ . \tag{3.16}$$

The equation for L_1 (3.10) can be solved now. The solution is (again) rather complicated, given in terms of gamma functions, but asymptoticly takes the form

$$L_1(r) = \frac{1}{\sqrt{aG}} \frac{e^{-\frac{r}{\sqrt{aG}}}}{r^{\frac{GM}{\sqrt{aG}}}} \ . \tag{3.17}$$

The last step before turning to numerical evaluation is to find exprasion for $\nu(r)$. If we integrate (3.2) we get

$$\lambda(r) + \nu(r) = -\delta \frac{\Gamma\left(1 - \frac{GM}{\sqrt{aG}}, \frac{r}{\sqrt{aG}}\right)}{(aG)^{1/2+\frac{GM}{2\sqrt{aG}}}} \ , \tag{3.18}$$

which we can solve asymptoticly to obtain

$$N_1(r) \simeq -\frac{e^{-\frac{r}{\sqrt{aG}}}}{r^{1+\frac{GM}{\sqrt{aG}}}} \ . \tag{3.19}$$

At this point we have enough information on the boundary conditions to carry a numerical analysis. I used Mathematica to numericaly solve (3.3) and (3.4) using the the above results as boundry conditions for $r \to \infty$. The resulting plots can be seen in figures 3 and 4.

If the numerical solution we found is indeed a viable solution for the whole space, than clearly something interesting has happened. First, outside $r = h$ the scalar field takes its regular form of reciprocal of G, but for $r < h$ its value seizes to be constant. Clearly enough, e^λ and e^ν also experiencing something interesting around $r = h$ as well. Thus, instead of recovering the RN solution, the $s \to 0$ solution admit RN structure only as an exterior for some new structure below $r < h$. Another interesting fact, is that there is no $t \leftrightarrow r$ flip. Since $r = h$ plays a crucial role in our new "would-have-been black hole" we will refer to it as the *would-have-been horizon*.

3.2 Next to the Origin: A Novel Core

We now focus our attention to the inside core, $r < h$, and repeat the procedure, this time using $r \to 0$ approximations as boundary conditions.

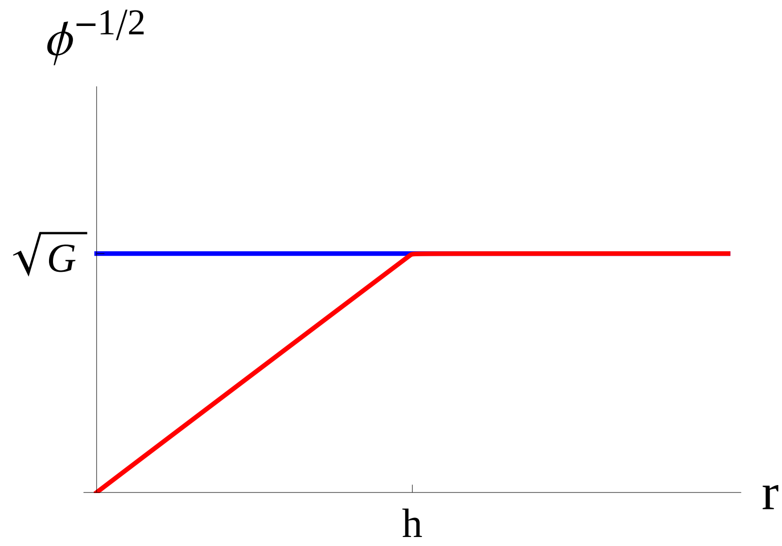

Figure 3: A numeric solution for the field equations using asymptotic boundary conditions with the constants values taken as $Q = G = 1; M = 1.25; s = 0.01$. As $s \to +0$, general relativity is recovered at the exterior region, but is spontaneously violated in the inner core.

The key for the analysis of this part, is to notice that our numeric solution in the previous section predicts that e^{λ} reduce to zero very fast below the would-have-been horizon. Admitting a solution with vanishing e^{λ}, eqs. (3.1) -(3.4) reduces to

$$\phi''(r) + \left(\frac{\phi''(r)}{\frac{2}{r}\phi(r) + \phi'(r)} - \lambda'(r)\right)\left(\phi'(r) - \frac{2}{r}\phi(r)\right) - \frac{2}{r^2}\phi(r) = 0 \ , \tag{3.20}$$

$$\phi''(r) + \phi'(r)\left(\frac{\phi''(r)}{\frac{2}{r}\phi(r) + \phi'(r)} - \lambda'(r) + \frac{2}{r}\right) = 0 \ . \tag{3.21}$$

We can extract a decoupled equation for ϕ

$$\frac{2\phi(r)}{r^2} - \frac{2\phi'(r)}{r} + \frac{2\phi(r)\phi''(r)}{r\phi'(r)} = 0 \ , \tag{3.22}$$

with the core solution being

$$\phi_c(r) = A_\phi \left(\frac{r}{h}\right)^{\epsilon - 2} \ , \tag{3.23}$$

where A_ϕ and ϵ are constants of integration, and h is just a scaling parameter for later

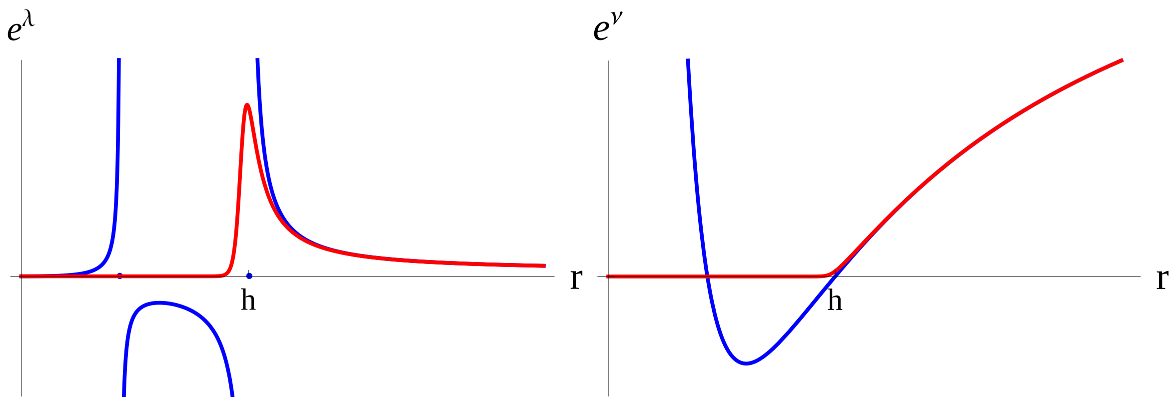

Figure 4: A generic $e^{\lambda(r)}$, $e^{\nu(r)}$ plots (red line). Whereas the exterior RN is recovered at the $s \to +0$ limit, the overall configuration conceptually differs from the full $s = 0$ RN solution (blue line).

purposes. It is now straightforward to find the core solutions for λ and ν:

$$\lambda_c(r) = 2\left(\epsilon - 3 + \frac{3}{\epsilon}\right) \mathrm{Log}\left(\frac{r}{h}\right) + \mathrm{Log}\left(A_\lambda\right) \ , \tag{3.24}$$

$$\nu_c(r) = 2\left(-2 + \frac{3}{\epsilon}\right) \mathrm{Log}\left(\frac{r}{h}\right) + \mathrm{Log}\left[A_\nu\right) \ . \tag{3.25}$$

A note is in order: The ϵ introduced here as a constant of integration, will later be interpreted as our small parameter. Numerical evaluation suggests that there is a linear connection between ϵ and the small parameter from the asymptotic expansion s given by the empirical formula [3]

$$\epsilon \simeq \frac{4e^{-\frac{h}{\sqrt{aG}}}s}{\left(1 - \frac{GQ^2}{h^2}\right) h^{1+\frac{GM}{\sqrt{aG}}}} \ . \tag{3.26}$$

As in the asymptotic case, its time to do a numerical evaluation for (3.3) and (3.4). Unlike the previous section, it turns out to be a problem. Instead, I used the boundary conditions to solve the following equations:

$$\phi''(r) \quad + \left(\frac{2}{r} + \frac{\nu'(r) - \lambda'(r)}{2}\right) \phi'(r) = 0 \ , \tag{3.27}$$

$$\phi''(r) \quad + \frac{1}{2}\left(\lambda'(r) - \nu'[r]\right) \left(\frac{2}{r}\phi(r) - \phi'(r)\right) - \frac{2}{r^2}\left(1 - de^{\lambda(r)}\right) \phi(r) = 0 \ . \tag{3.28}$$

Here we have neglected all terms proportional to e^λ in eqs. (3.27) and (3.4), beside the one multiplied by d. The results can be seen in figure 5.

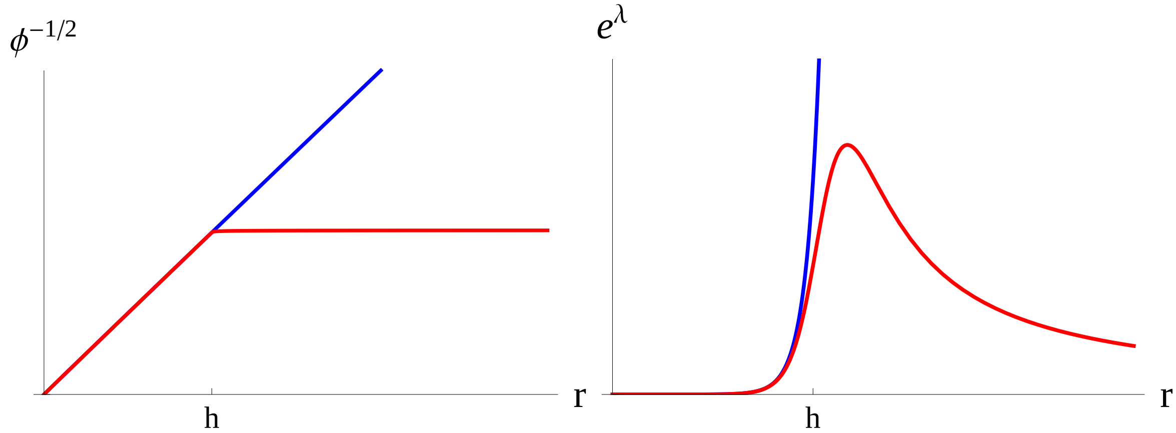

Figure 5: The inner properties of the solution, such as the varying Newton's constant and the suppression of $e^{\lambda(r)}$, are fully captured by the $d = 0$ approximation (blue). ϕR gravity, switched on by $d = 1$, already exhibits the transition profile (red).

The suppression of e^λ in the inner core is fully captured by the $d = 0$ approximation. Switching to $d = 1$, the asymptotic solution is recovered, enabling us to find the structure of the phase transition. Note that the structure of the core remains the same even for $d = 0$, which tells us something important about the inside solution: it knows nothing about the potential, nor about the electromagnetic energy momentum contributions. This continues to be true even with $d = 1$, which correspond to plain ϕR gravity. The influence of the different contributions on the core solution is determined by matching with the exterior, but the structure of the interior is expected to remain the same. We could have chosen Swartzschild or Kerr as our exterior without effecting the interior too much.

3.3 Connecting Two Regions: The Would-Have-Been Horizon

We have solved for $r \gg h$ and $r \ll h$. Now it's time to connect between the known RN exterior and the new intriguing interior. Actually, the matching for ϕ is straightforward:

after inspecting the appropriate graphs, we set the coefficient of eq. (3.23) A_ϕ as G^{-1} so that

$$\phi(r) \approx \frac{1}{G}\left(\frac{h}{r}\right)^{2-\epsilon} , \tag{3.29}$$

will take the value of the reciprocal of G at $r = h$.

The rest of the matching is not so simple, and again we will use our preliminary numerical insight to find expressions for the transition profile. Appropriate approximation leads us to the following equations (for detailed information, see appendix D)

$$r - \bar{r} = -\frac{h}{p}\left(\frac{AG}{2h}\right)^2 (x - \mathrm{Log}[1+x]) , \tag{3.30}$$

$$e^{\lambda(r)} = -\frac{p}{1-\frac{GQ^2}{h^2}}\left(\frac{2h}{AG}\right)^2\left(\frac{1+x}{x^2}\right) , \tag{3.31}$$

$$e^{\nu(r)} = -\frac{p}{1-\frac{GQ^2}{h^2}}(1+x) . \tag{3.32}$$

where p and A are all constants of integration, and

$$x = \frac{2h}{AG}e^{\frac{\nu(r)-\lambda(r)}{2}} . \tag{3.33}$$

The asymptotic solution will respond to $x \to \infty$ and the core solution to $x \to -1^+$.

3.3.1 Matching

Let us begin with the asymptotic matching. Outside the horizon we want our solution to look like the RN solution:

$$e_{RN}^{-\lambda(r)} = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \approx \frac{2G}{h^2}\left(M - \frac{Q^2}{h}\right)(r-h) = \frac{1}{h}\left(1 - \frac{GQ^2}{h^2}\right)(r-h) , \tag{3.34}$$

where in the last step I used the fact that $\frac{GM}{h} = \frac{1}{2}\left(1+\frac{GQ^2}{h^2}\right)$. Expanding the horizon solution for $x \to \infty$ eq. (3.30) takes the form

$$r - \bar{r} \approx -\frac{1}{hp}\left(\frac{AG}{2}\right)^2 x . \tag{3.35}$$

We keep in mind, that even though $x \to \infty$, x/p remains small, in order that $r - \bar{r}$ will remain small (this observation will be clear later, when we identify p as order of $1/\epsilon$). Substituting into the expansion of e^λ and e^ν gives

$$e^{\lambda(r)} \approx -\frac{ph^2}{1-\frac{GQ^2}{h^2}}\left(\frac{2}{AG}\right)^2\frac{1}{x} = -\frac{h}{1-\frac{GQ^2}{h^2}}\frac{1}{r-\bar{r}} \ , \tag{3.36}$$

$$e^{\nu(r)} \approx -\frac{p}{1-\frac{GQ^2}{h^2}}x = \frac{p^2}{h}\left(\frac{1}{1-\frac{GQ^2}{h^2}}\right)\left(\frac{2h}{AG}\right)^2(r-\bar{r}) \ . \tag{3.37}$$

The former agrees with the RN solution for $\bar{r} = h$. Comparing the latter with the outer horizon approximated RN solution we have

$$\frac{p}{A} = \frac{G}{2h}\left(1-\frac{GQ^2}{h^2}\right) \ . \tag{3.38}$$

For the interior matching, we would want our solution to match to the core solution obtained in section 3.2

$$e^{\lambda(r)} = A_\lambda\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6+2\epsilon} \ , \tag{3.39}$$

$$e^{\nu(r)} = A_\nu\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4} \ . \tag{3.40}$$

We set $x = -1+\delta$ where $\delta \to 0$ and expand (3.30) to obtain

$$r - h \approx \frac{h}{p}\left(\frac{AG}{2h}\right)^2 \log\delta \ . \tag{3.41}$$

Keeping in mind that $r \to h$

$$\log\left(\frac{r}{h}\right) = \log\left(1+\frac{r-h}{h}\right) \approx \frac{r-h}{h} = \frac{1}{p}\left(\frac{AG}{2h}\right)^2\log\delta \ , \tag{3.42}$$

so that

$$\delta = \left(\frac{r}{h}\right)^{\frac{4h^2p}{A^2G^2}} \ . \tag{3.43}$$

Using this for the approximation of $e^{\nu(r)}$ yields

$$e^{\nu(r)} \approx -\frac{p}{1-\frac{GQ^2}{h^2}}s = -\frac{p}{1-\frac{GQ^2}{h^2}}\left(\frac{r}{h}\right)^{\frac{4h^2p}{A^2G^2}} \ , \tag{3.44}$$

and comparing it with the first order of (3.40) to identify p

$$p = \frac{\epsilon}{6}\left(1-\frac{GQ^2}{h}\right)^2 \ . \tag{3.45}$$

3.3.2 Phase Transtion

Finally, we are done with the matching procedure and defined our metric for the entire space. The exterior recovered the known RN metric, while the interior exhibits new geometry that will be discussed in the next chapter. Those two regions are connected by a transition profile which, as we will now show, can characterize the phase transition of our theory.

If we set $\epsilon = 0$, the transition profile is restricted to a two-sphere with $r = h$. In that case, the scalar field and the metric coefficients seize to be smooth. On the other hand, if we take $\epsilon \to 0$, the profile gains width and the associated functions regain smoothness. We can characterize that behaviour by inspecting the behaviour of the metric functions. Such features are demonstrated in figure 6 for the radial coefficient.

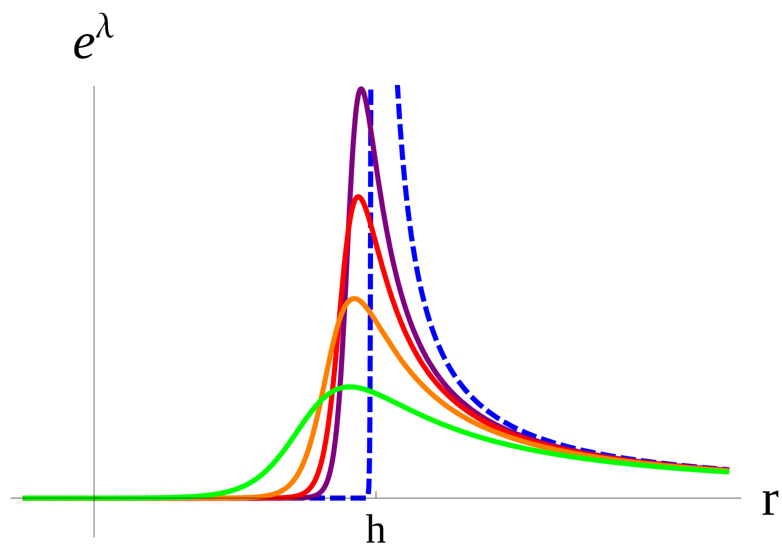

Figure 6: The in-out transition profile is plotted for a decreasing series of ϵ values. The phase transition into the exterior Reissner Nordstrom solution (dashed line) occurs as $\epsilon \to +0$.

Let the maximum of $e^{\lambda(r)}$ serve as the characteristic cut-off for the spontaneously induced general relativity. From eq. (3.31) we find that the maximum value occurs for $x = -2$ and takes the value

$$e^{\lambda}_{max} = \frac{3}{2\epsilon\left(1 - \frac{GQ^2}{h^2}\right)} \,. \tag{3.46}$$

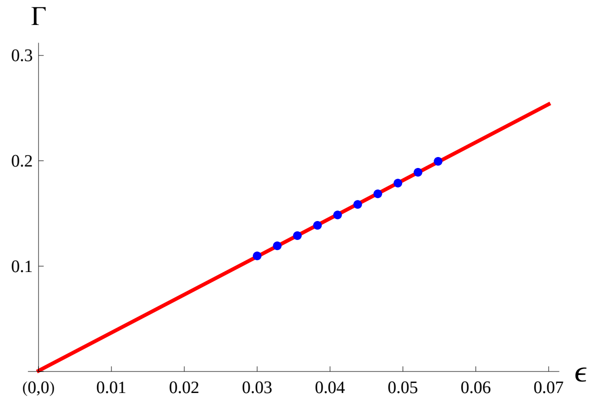

Figure 7: The width of the transition profile is proportional to ϵ. For $\epsilon \to 0$ the profile is just a two-sphere of radius h.

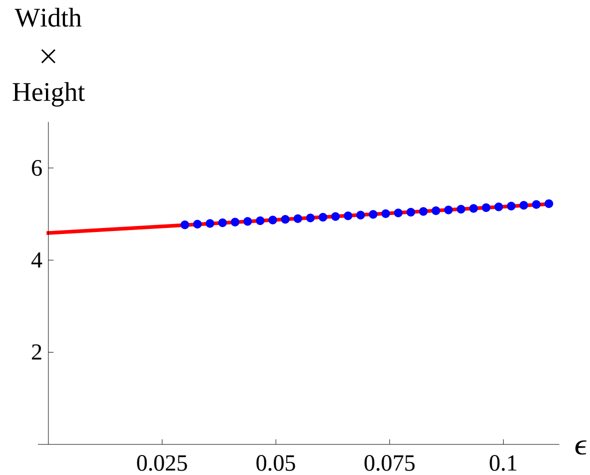

Figure 8: $e^{\lambda}_{max}\Gamma$ is plotted for a decreasing series of ϵ values. We set $\left(1 - \frac{GQ^2}{h^2}\right) = 100$ and from the intersection with $\epsilon = 0$ we can extract the dependency $\Gamma(\epsilon)$.

An analytical expression for the typical width Γ (where $e^{\lambda(r)}$ drops to half its maximal size) is not so simple to obtain. Instead we can use the numerical plots to examine its features (see figure 7). The crucial point is, that for small enough values of ϵ, the transition profile acts as δ-function, and $e^{\lambda}_{max}\Gamma$ becomes constant. Using the constant width-height value, we can deduce that Γ is proportional to the small parameter, $\Gamma = kh\epsilon$, with the ratio taking the empirical value $k = \frac{1}{60}$.

4 The Inner Metric

Let us write down the line element in the core, which we worked so hard to obtain

$$\mathrm{ds}_{\mathrm{in}}^2 \approx -\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}\mathrm{dt}^2+\frac{\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6+2\epsilon}}{\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}}\mathrm{dr}^2+r^2\mathrm{d}\Omega^2 \ . \tag{4.1}$$

This is accompanied by the associated scalar field

$$\phi(r) \approx \frac{1}{G}\left(\frac{h}{r}\right)^{2-\epsilon} \ . \tag{4.2}$$

At this point, I would like to discuss some of the geometrical features of this metric. Naturally, we will be interested in the $\epsilon \to 0$, since this is the limit where the core connects to the RN exterior at the would-have-been horizon.

4.1 Vanishing Volume

A simple calculation will reveal a unique feature of this metric. The spatial volume of a sphere with circumferential radius r is given by

$$V(r) = \int_\Sigma d^3x\sqrt{\gamma^{(3)}} = 4\pi\int_0^r e^{\frac{1}{2}\lambda(r')}r'^2dr' \ , \tag{4.3}$$

and for $r < h$ the volume is given by

$$\begin{aligned} V(r) &= 4\pi\left(1-\frac{GQ^2}{h^2}\frac{\epsilon}{6}\right)^{-1/2}\int_0^r\left(\frac{r'}{h}\right)^{\frac{3}{\epsilon}-3+\epsilon}r'^2dr' \\ &= 4\pi h^3\left(1-\frac{GQ^2}{h^2}\frac{\epsilon}{6}\right)^{-1/2}\frac{\epsilon}{3+\epsilon^2}\left(\frac{r}{h}\right)^{\frac{3}{\epsilon}+\epsilon} \\ &\approx \frac{4\pi h^3}{3}\sqrt{\frac{6\epsilon}{1-\frac{GQ^2}{h^2}}} \ . \end{aligned} \tag{4.4}$$

Thus, the volume of sphere with radius r, vanishes for every $r < h$! Note that for each sphere the surface area remain the same $S(r) = 4\pi r^2$. That feature suggest a possible reason for question why is the black hole entropy formula (1.17) proportional to the surface area of the system, rather than the volume.

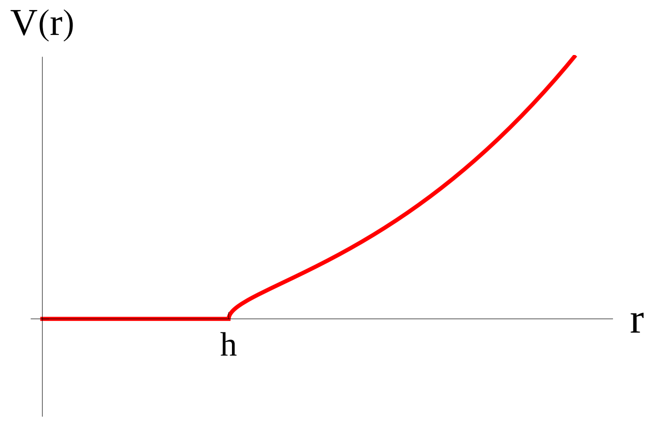

Figure 9: The invariant volume $V(r)$: Every inner concentric sphere of finite surface area $4\pi r^2$ exhibits a vanishingly small volume.

4.2 Light Cones

We will continue our analysis of the metric with light cones. The light cone is defined by

$$\frac{dr}{dt} = e^{\frac{\nu(r)-\lambda(r)}{2}} \ , \tag{4.5}$$

which is the same as posing $ds^2 = 0$ at the metric (2.5). For the exterior one have

$$\frac{dr}{dt} = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) \ , \tag{4.6}$$

and next to the horizon $r = h + \delta r$

$$\frac{dr}{dt} \approx \frac{2G}{h^2}\left(M - \frac{Q^2}{h}\right)\delta r = \left(1 - \frac{GQ^2}{h^2}\right)\frac{\delta r}{h} \ . \tag{4.7}$$

For the inside metric (4.1)

$$\frac{dr}{dt} = \left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{1-\epsilon} \ , \tag{4.8}$$

since ϵ is very small, we recast the light cone equation into

$$\frac{dr}{dt} \approx \left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon r}{6h} \ . \tag{4.9}$$

From comparing between inner and outer light cone structure, we see that the outside $\delta r \to 0$ role, is played inside by $\epsilon \to 0$. This is a crucial feature of the light cones, since we

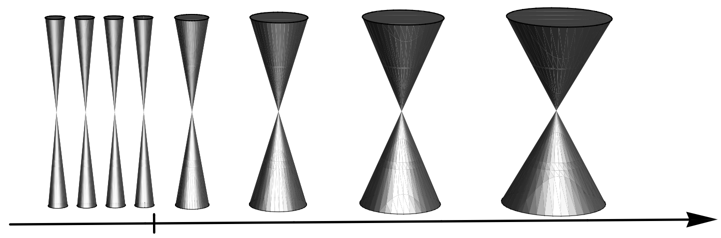

Figure 10: In natural coordinates, the light cones appear to close up as we approach $r = h$, just as ordinary black hole. Below the horizon they remain closed, as if the horizon is stretched all the way to the origin.

can see that the light cones inside are closing for every $r < h$, the same way they are closing for an observer who is falling toward the outer horizon of a regular RN black hole. Thus, below the horizon a layer upon layer of horizons are being formed.

A nice example will be to calculate how much time Δt does it take light to fall into the center. Integrating (4.9) yields

$$\ln\left(\frac{r}{r_0}\right) = -\frac{\epsilon}{6h}\left(1 - \frac{GQ^2}{h^2}\right)\Delta t \ , \tag{4.10}$$

so the time

$$\Delta t = \frac{h^3}{h^2 - GQ^2}\frac{6}{\epsilon}\ln\left(\frac{r_0}{r}\right) \ , \tag{4.11}$$

gets infinitely large as ϵ gets infinitely small.

4.3 Geodesics

We now turn to compute time-like geodesics for the inside metric. For a massive particle we know that

$$-\left(\frac{ds_{in}}{d\tau}\right)^2 = 1 = \left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}\dot{t}^2 - \frac{\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6+2\epsilon}}{\left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin(\theta)^2\dot{\phi}^2\right) \ . \tag{4.12}$$

Now, if K^μ is a killing vector, then we know that

$$K_\mu\frac{dx^\mu}{d\tau} = \text{constant} \ . \tag{4.13}$$

Using the timelike killing vector $K^\mu = (\partial_t)^\mu = (1,0,0,0)$ we obtain the first constant of motion

$$\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}\dot{t}=\varepsilon \ . \tag{4.14}$$

Setting the motion on a plane $\theta=\pi/2$ and using the azimuthal killing vector $R^\mu = (\partial_t)^\varphi = (0,0,0,1)$ we have a second constant of motion

$$r^2\dot{\varphi}=J \ , \tag{4.15}$$

and equation (4.12) is simplified to

$$1=\frac{\varepsilon^2}{\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}}-\frac{\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6}}{\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}}\dot{r}^2-r^2\left(\frac{J}{r^2}\right)^2 \ . \tag{4.16}$$

Replacing the proper time derivative with the time coordinate derivative

$$\dot{r}^2=\left(\frac{dr}{d\tau}\right)^2=\left(\frac{dt}{d\tau}\right)^2\left(\frac{dr}{dt}\right)^2=\frac{\varepsilon^2}{\left(\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}\right)^2}\left(\frac{dr}{dt}\right)^2 \ , \tag{4.17}$$

and manipulating the equation yields an equation for radial velocity

$$\left(\frac{dr}{dt}\right)^2=\left(\frac{\epsilon}{6}\right)^2\left(1-\frac{GQ^2}{h^2}\right)^2\left(\frac{r}{h}\right)^{2-2\epsilon}-\frac{1}{\varepsilon^2}\left(\frac{\epsilon}{6}\right)^3\left(1-\frac{GQ^2}{h^2}\right)^3\left(\frac{r}{h}\right)^{-2+\frac{6}{\epsilon}-2\epsilon}\left(1+\frac{J^2}{r^2}\right) \ , \tag{4.18}$$

which go to zero for $\epsilon\to 0$. The above equation recover the equation light-like geodesic equation (4.9) for $\varepsilon\to\infty$ as required.

The equation for the angular velocity is much easier to obtain, and it is straight forward to derive it from the killing vectors eqs. (4.14) and (4.15):

$$\frac{d\varphi}{dt}=\frac{d\varphi}{d\tau}\frac{d\tau}{dt}=\frac{1}{\varepsilon}\frac{\epsilon}{6}\left(1-\frac{GQ^2}{h^2}\right)\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4}\frac{J}{r^2} \ . \tag{4.19}$$

We will end our discussion on geodesics with an equation for a trajectory $r(\varphi)$. With the help of (4.18) and (4.19) we can write

$$\left(\frac{dr}{d\phi}\right)^2=\frac{\left(\frac{dr}{dt}\right)^2}{\left(\frac{d\phi}{dt}\right)^2}=\frac{\left(\frac{r}{h}\right)^{2-2\epsilon}-\frac{1}{E^2}\left(\frac{\epsilon}{6}\right)\left(1-\frac{GQ^2}{h^2}\right)\left(\frac{r}{h}\right)^{-2+\frac{6}{\epsilon}-2\epsilon}\left(1+\frac{J^2}{r^2}\right)}{\left(\frac{1}{\varepsilon}\left(\frac{r}{h}\right)^{-4+\frac{6}{\epsilon}}\frac{J}{r^2}\right)^2} \ . \tag{4.20}$$

Dropping the second term in the nominator (for small ϵ) and taking the square root

$$\frac{dr}{d\varphi} = \frac{h^2\varepsilon}{J}\left(\frac{r}{h}\right)^{7-\frac{6}{\epsilon}-\epsilon} \ , \tag{4.21}$$

with the solution

$$\Delta\varphi = \frac{J^2\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6+\epsilon}}{h\left(-6+\frac{6}{\epsilon}+\epsilon\right)\varepsilon^2} \approx \frac{J}{h\varepsilon}\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-6} \ , \tag{4.22}$$

so the trajectory is

$$r(\varphi) = \left(\frac{h\varepsilon}{J}\frac{6}{\epsilon}\Delta\varphi\right)^{\frac{1}{\frac{6}{\varepsilon}-6}} \ . \tag{4.23}$$

One can see, that falling from h to the center will cover an angle of

$$\Delta\varphi = \frac{J}{h\varepsilon}\frac{\epsilon}{6} \ . \tag{4.24}$$

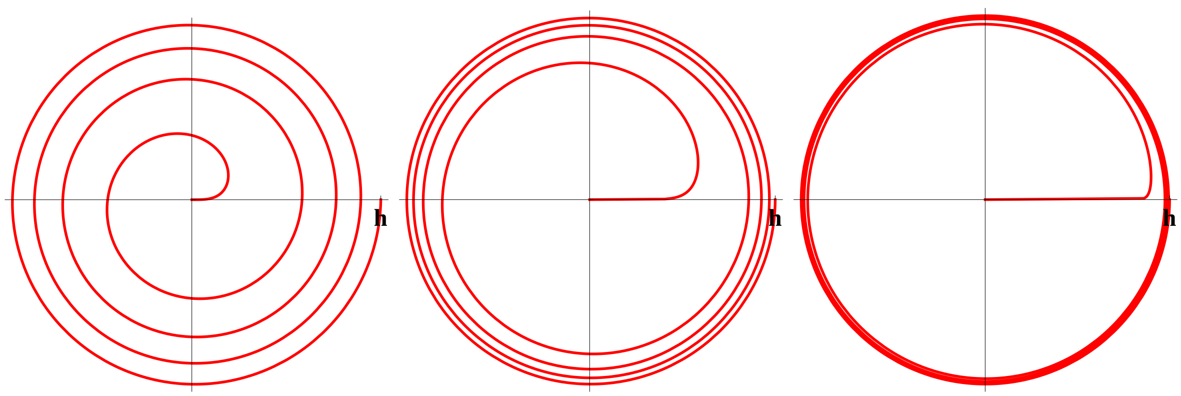

Figure 11: $r(\varphi)$ trajectory: each of the trajectories cover the same solid angle (8π), with decreasing values of ϵ.

4.4 The Origin

To gain some more insight into the inner metric, one may define a proper length by

$$d\eta = \sqrt{\frac{6}{\left(1-\frac{GQ^2}{h^2}\right)\epsilon}}\left(\frac{r}{h}\right)^{\frac{3}{\epsilon}-3+\epsilon} dr \ , \tag{4.25}$$

with

$$\eta = \frac{\sqrt{\frac{6}{\left(1-\frac{GQ^2}{h^2}\right)\epsilon}}h}{\frac{3}{\epsilon}-2+\epsilon}\left(\frac{r}{h}\right)^{\frac{3}{\epsilon}-2+\epsilon} \quad . \tag{4.26}$$

The inner metric (4.1) in terms of η is

$$\begin{aligned} ds^2_{in} = & - \quad \left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{\left(\frac{3}{\epsilon}-2+\epsilon\right)^2}{\frac{6}{\left(1-\frac{GQ^2}{h^2}\right)\epsilon}}\eta^2\right)^{\frac{\frac{6}{\epsilon}-4}{\frac{6}{\epsilon}-4+\epsilon}} dt^2 + d\eta^2 \\ & + \quad h^2\left(\frac{1}{6}\left(1-\frac{GQ^2}{h^2}\right)\left(\frac{3}{2\epsilon}-2+\frac{5\epsilon}{3}-\frac{2\epsilon^2}{3}+\frac{\epsilon^3}{6}\right)\eta^2\right)^{\frac{1}{\frac{3}{\epsilon}-2+\epsilon}} d\Omega^2 \ , \end{aligned} \tag{4.27}$$

expanding up to order of $\mathcal{O}(\epsilon)$ a Rindler structure emerges

$$ds^2_{in} = -\kappa^2\eta^2 dt^2 + d\eta^2 + h^2 d\Omega^2 \ , \tag{4.28}$$

with κ being

$$\kappa = \frac{1}{2h}\left(1^2 - \frac{GQ^2}{h^2}\right) \ , \tag{4.29}$$

with a Rindler horizon at $\eta = 0$.

It is interesting to compare that result with the RN case. In the RN case, we can derive Rindler structure as well, (with the same κ). However, the proper length will be defined by

$$\eta = \sqrt{\frac{2h(r-h)}{\left(1-\frac{GQ^2}{h^2}\right)}} \ . \tag{4.30}$$

The crucial point is that $\eta = 0$ in the RN metric corresponds to $r = h$, while in our case it correspond to $r = 0$. Taking this point of view, we can say that in some sense *the horizon shifted its place to the origin.*

Another key feature of the theory has to do with κ itself, the surface gravity function

$$\kappa(r) = \left(e^{\frac{\nu}{2}}\right)' e^{-\frac{\lambda}{2}} \ , \tag{4.31}$$

where usually the surface gravity is computed at the horizon. For The RN metric, the surface gravity function is

$$\kappa(r) = \frac{G}{r^2}\left(M - \frac{Q^2}{r}\right) \ , \tag{4.32}$$

which is the same as (4.29) only for $r = h$. Surprisingly, in our case

$$\kappa(r) = \frac{G}{h^2}\left(M - \frac{Q^2}{h}\right) = \text{CONST} \ , \tag{4.33}$$

for every $r < h$ all the way to the origin. On contrary to the *shifted horizon*, in that sense the entire core functions as *stretched horizon*, which will have far reaching consequences on the mass and thermodynamics of the black hole.

With that preface, we can now discuss the origin itself. From (4.27) we can calculate Kretschmann scalar. The exact result is too long to write down, but it is interesting to inspect the expression for two different limits. Taking ϵ to 0 we get

$$\lim_{\epsilon \to 0} \mathcal{K} = \frac{4}{h^2} \ , \tag{4.34}$$

the same as RN (1.7). On the other hand, taking first the limit of $\eta \to 0$ and than keeping only the leading order term for ϵ, than the most relevant term is

$$\mathcal{K} = \frac{16\epsilon^2}{9\eta^4} \ , \tag{4.35}$$

which is singular at the origin. It's not just Kretschmann scalar, the same thing happens with others scalars in the theory, such as Ricci's, indicating that two different limits leading us to two different results, one of them gives rise to a singular origin while the other is completely regular.

A possible singularity in the origin is somewhat problematic. No actual horizon protect the singularity and therefore we have a naked singularity, which is not so popular among physicist. There are few possibilities addressing that issue

(a) As we have pointed out, the area between the would-have-been horizon and the origin serves as a pseudo horizon (4.9) that give some "protection" to the singularity. Yet, a patient observer is able to see unbounded high curvature. This option make this model relevant only in specific cases, such as black hole that is reminiscent of some early universe.

(b) It may be possible to "cure" the singularity by invoking a more complicated Lagrangian or introducing some quantum effects.

(c) A possible argument in the favour of the ϵ limit is that we can't treat η and ϵ on the same footing [3]. Since η is a coordinate, and ϵ is a parameter specifying the metric, we should keep ϵ small in respect with the metric at any given point in space. In other words, we demand that $\epsilon \ll \eta^2$, which will cause our scalars to converge at the origin.

4.5 Einstein frame

In the Lagrangian (2.1) we have coupled the scalar field to Ricci scalar only, rather then both Ricci scalar and the electromagnetic tensor, which defined Jordan frame as our physical one. We can find the metric in Einstein frame using the conformal transformation

$$\tilde{g}_{\mu\nu} = \phi g_{\mu\nu} = \frac{1}{G}\left(\frac{r}{h}\right)^{-2+\epsilon} , \tag{4.36}$$

the new line element will be of the form

$$d\tilde{s}^2 = -\left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-2-\epsilon}\mathrm{dt}^2 + \frac{\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-4+\epsilon}}{\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}}\mathrm{dr}^2 + \left(\frac{r}{h}\right)^{2-\epsilon} r^2 d\Omega^2 \ . \tag{4.37}$$

By an accompanying change of variables, namely by

$$\rho = r\left(\frac{r}{h}\right)^{1-\frac{\epsilon}{2}} , \tag{4.38}$$

the resulting metric $d\tilde{s}^2_{in}$ we obtain

$$d\tilde{s}^2_{\mathrm{in}} = -\left(1 - \frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}\left(\frac{\rho}{h}\right)^{\frac{3}{\epsilon}-\frac{1}{4}}\mathrm{dt}^2 + \frac{\left(\frac{\rho}{h}\right)^{3\left(\frac{1}{\epsilon}-\frac{3}{4}\right)}}{4\left(1-\frac{GQ^2}{h^2}\right)\frac{\epsilon}{6}}d\rho^2 + \rho^2 d\Omega^2 \ . \tag{4.39}$$

The resemblance between the two frames line element, assure us the forthcoming results will be frame independent.

5 Properties of the would-have-been Black Hole

5.1 Mass

Following our discussion from the introduction (section 1.3), we should choose what definition we want to use in our calculations. The ADM mass in our case will obviously give the RN ADM mass at infinity M. For a more "local" definition, we should use the Komar mass. Using equation (1.16) outside of the would-have-been horizon yields the known RN result

$$m_k^{out}(r) = M - \frac{Q^2}{r} \ . \tag{5.1}$$

On the other hand, calculating Komar integral in the interior region gives

$$m_k^{in}(r,\epsilon) = -\frac{r^2\left(1-\frac{GQ^2}{h^2}\right)\left(\frac{r}{h}\right)^{-\epsilon}(-3+2\epsilon)}{6Gh} \ , \tag{5.2}$$

and for zero order in ϵ

$$m_k^{in}(r) = \left(1-\frac{GQ^2}{h^2}\right)\frac{r^2}{2Gh} = \left(M-\frac{Q^2}{h}\right)\frac{r^2}{h^2} \ . \tag{5.3}$$

The Komar mass function is plotted in figure 12

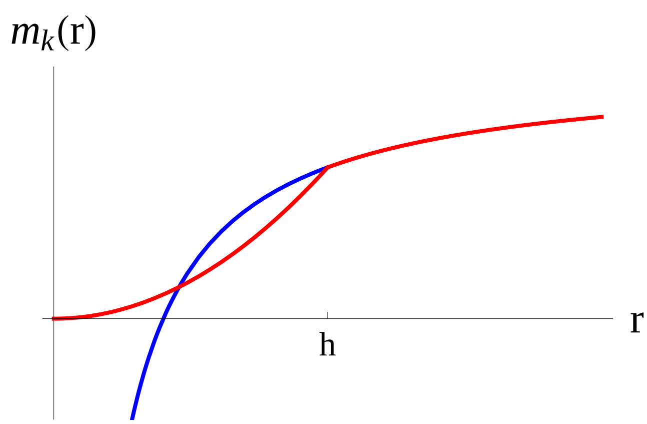

Figure 12: Unlike in the Reissner Nordstrom case (blue line), the Komar mass $m_K(r)$ is non-singular at the origin, and furthermore obeys the positive energy condition.

Note that for RN case, result (5.1) hold everywhere. Eq. (5.3) offers us a new way for the mass distribution in the black hole. Instead of a singular mass at the origin, now every sphere bellow $r = h$ holds mass proportional to its surface area

$$m_k(r) = m_k(h)\frac{r^2}{h^2} \ . \tag{5.4}$$

Another encouraging feature is that the mass is definite non-negative, obeying positive energy condition for every r.

5.2 Charge

The electric field is given in equation (3.1). It is straightforward to derive the corresponding electric field:

$$E_{out}(r) = \frac{Q}{r^2} \tag{5.5}$$

$$E_{in}(r) = \frac{Q}{h^2}\left(\frac{r}{h}\right)^{\frac{6}{\epsilon}-7+\epsilon} \tag{5.6}$$

Despite the peculiar form of the electric field, there is nothing new here. If we use gauss law we can see that the charge is concentrated in the origin. The total charge can be calculate as a boundary integral [4]

$$\mathcal{Q}(r) = -\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu F^{\mu\nu} \ . \tag{5.7}$$

the boundry $\partial\Sigma$ is of a two-sphere with metric $\gamma_{ij}^{(2)}$ and outward-pointing normal vector σ^μ. n^μ is the unit normal vector associated with Σ. The normal vectors, normalized to $n_\mu n^\mu = -1$ and $\sigma_\mu\sigma^\mu = 1$, have nonzero components

$$n_t = -e^{\frac{\nu}{2}} \ , \tag{5.8}$$

$$\sigma_r = e^{\frac{\lambda}{2}} \ , \tag{5.9}$$

so the total charge is (remember that $F^{tr} = g^{tt}g^{rr}F_{tr}$ and that $\sqrt{\gamma^{(2)}} = r^2\sin\theta$)

$$\mathcal{Q}(r) = -\frac{1}{4\pi}\int_{\partial\Sigma}(-)e^{\frac{\nu}{2}}e^{\frac{\lambda}{2}}(-)e^{-\nu}e^{-\lambda}\frac{Q}{r^2}e^{\frac{\lambda(r)+\nu(r)}{2}}r^2\sin\theta d\theta = Q \ . \tag{5.10}$$

this result is independent of r, and therefore we deduce that for every $r > 0$ the charge inside is Q.

5.3 Thermodynamics

The last sections gave us a few hints about new physics in the region below the would-have-been horizon. Closing light cone structure, constant surface gravity and stretched non-singular mass all indicate that, in some sense, the origin has taken the place of the horizon. Indeed, the Rindler structure was recovered for the origin rather then in the horizon. We will conclude the characterization of the black hole by discussing the implications of the above features on the thermodynamics. For example, we know that the temperature of the black hole, as seen by observer in infinity is

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi h}\left(1 - \frac{GQ^2}{h^2}\right) \ . \tag{5.11}$$

Unlike conventional physics, κ remain the same all the way to the origin, rather only on the horzion.

Inspired by Smarr formula [23], we can combine Hawking temperature with the mass equation (5.3) on the horizon to write

$$m(h) = M - \frac{Q^2}{h} = 2T_H S(h) \ , \tag{5.12}$$

which gives rise to the known formula for the black hole entropy

$$S(h) = \frac{\pi h^2}{G} \ . \tag{5.13}$$

Motivated by the core temperature formula, we may ask ourself why stop at the horizon? Multiplying by r^2/h^2 will lead us to

$$m_{in}(r) = 2T_H S(r) \ , \tag{5.14}$$

so that the entropy

$$S(r) = S(h)\frac{r^2}{h^2} = \frac{\pi r^2}{G} \ , \tag{5.15}$$

is distributed, like the mass, on the entire core. Note that an analogous formula simply does not exist for the interior of an RN black hole.

The source of the entropy packing formula can be traced back to the varying Newton's constant. To understand this point better, let consider an uncooperative physicist who insist that Newton's constant is just a constant. since that in our model, "Newton's constant" inside the core take the form of

$$G_{in}(r) = G\frac{r^2}{h^2} \ . \tag{5.16}$$

The stubborn physicist will regain his general relativity back by making the effective replacement

$$G \to G\frac{h^2}{r^2} \ . \tag{5.17}$$

The Hawking temperature is defined asymptotically, and as such is fully respected by our 'naive' observer. This will require transformation rules for the mass and charge

$$M \to M\frac{r^2}{h^2} \ , \tag{5.18}$$

$$Q \to Q\frac{r}{h} \ . \tag{5.19}$$

All the above nicely converge now back to

$$S = \frac{\pi h^2}{G} \to \frac{\pi r^2}{G} \ , \tag{5.20}$$

as it should be.

We now can identify our inner core as a onion-like structure [2]. Each "onion" shell with surface area $A(r) = 4\pi r^2$ and a vanishing volume (Eq. 4.4) contain mass and entropy proportional to its surface area. The last point we would like to clear out is how exactly the above configuration changes upon supplementing the pair M, Q by tiny amounts $\Delta M, \Delta Q$ respectively. As we mentioned earlier this question is meaningless inside the core of a RN black hole, but we do know that the horizon will be shifted by $h \to h + \Delta h$, which is nothing but the first law of black hole thermodynamics (1.18). The existence of the onion-like quantities such as the Komar mass and the entropy is then translated to the fact that each concentric sphere of radius r is puffed up to a new radius $r \to r + \Delta r$. in such way that

$$\Delta\left(\frac{r}{h}\right) = 0 \ , \tag{5.21}$$

due to the fact that the only length scale in the core is h. The new configuration is therefore nothing but a stretched version of the older configuration.

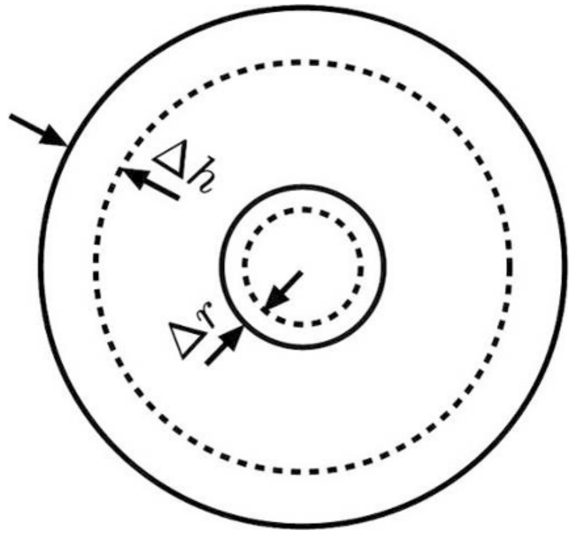

Figure 13: The $\{M+\Delta M, Q+\Delta Q\}$ configuration (solid circles) is a linearly stretched version of the $\{M, Q\}$ configuration (dashed circles). As $h \to h+\Delta h$, each point at a circumferential radius r gets radially shifted by an amount $\Delta r = \dfrac{r}{h}\Delta h$.

6 Summary and Outlook

In this thesis we presented a theoretical model of a spontaneously General Relativity coupled to electromagnetism, focusing the black hole limit. After introducing the action which governs the theory, we derived the associated gravitational and scalar field equations. We would like to emphasize that though some specific assumptions were made for the sake of clarity, the results obtained are general in the sense that

- They are not sensitive to the exact shape of the scalar potential, leaving the door open to a more general class of $f(R)$ gravity models,
- Einstein/Jordan frame independence, and,
- Brans-Dicke kinetic term is optional with only minor modifications

Thus, the model is generic and can be extended to a large class of gravitational theories theories.

Tagging the deviation of the results from the general relativistic solution with some ϵ, in such a way that $\epsilon = 0$ correspond to Reissner Nordström geometry, we have obtained the $\epsilon \to 0$ solution. The emerging solution admits a general relativistic limit only outside the *would-have-been horizon*, with the scalar field acquiring the vacuum expectation value of perturbed reciprocal Newton's constant. The large distance Yokawa perturbation magnitude is proportional to ϵ as well.

Unlike the exterior, the core of the black hole is dominated by a local varying scalar field, and as such exhibits new features. A smooth transition profile connects the interior and the exterior regimes at the would have been horizon; it's width is proportional to ϵ. For $\epsilon = 0$ the transition profile cease to be smooth, meaning that the various functions on the horizon develop asymptote or that their derivatives become discontinues for and a phase transition occurs.

The core of the black hole demonstrated unique properties, the first of them is the varying Newton's constant and the absence of signature flip upon crossing the would have been horizon. We have seen that the new core in some sense resembles a *maximally stretched horizon* - an area with vanishing volume and some of the characteristic of the horizon such as the closing light cones and surface gravity are kept "frozen" all the way to the origin. The near-horizon Rindler structure is shifted to the origin and together with the rest of the features of the core geometry provides some protection for the underlying singularity, Though the question of the singularity itself remains open.

The new physics associated with the inner core introduced us to a non-singular mass distributed along the entire core along with the entropy that is no more spread solely on the horizon. The overall picture is then of an onion-like entropy packing shell model, with the entropy of any inner sphere, being geometric in nature, is maximally packed and unaffected by the outer layers. Any additional entropy is maximally packed on its own external layer, with the overall mass being adjusted accordingly. the stack ed entropy configuration gives local realization of the 't Hooft-Susskind-Bousso holographic principle [24] [25] (the holographic bound, as we recall, is not applicable inside ordinary black holes).

The inclusion of electromagnetism as supported the generality of the maximally packing idea, and exactly the same structure is expected to hold once the cosmological constant, angular momentum and/or arbitrary number of dimensions enters the game since as we seen, the core geometry is mainly effected by the varying Newton's constant, and became aware of all other effects only through the matching procedure.

6.1 Outlook on Further Research

- Finite ϵ:

  Classically, with general relativity so well established, $\epsilon \to +0$ is indeed the limit to study. However, having quantum mechanics in mind, and appreciating the fact that the singularity at the origin will eventually be disarmed quantum mechanically, it is quite appealing to imagine a very small yet a finite ϵ. For example, the invariant width of the transition region may be fixed by the Planck length, namely $\sqrt{\epsilon}h \simeq \ell_{Pl}$.

- AdS/CFT correspondence:

The standard picture od the AdS/CFT correspondence relies on 5 dimensional Anti de Sitter with the horizon being a 3 dimensional flat space [26]. The deviations from the black hole solution are parametrized by tiny (local) boost parameters. It will be interesting to check the implication of adding an ϵ perturbation as well.

- Adding quadratic terms for the Lagrangian:

  The ϕR gravity is equivalent to $\alpha R + \beta R^2$ theories. We can try to add other forms of square terms of the curvature such as $R^{\mu\nu} R_{\mu\nu}$, or make use of Weyl tensor C^2. A Gauss-Bonnet term can also provide an interesting option, since by being a full derivative he contributes nothing to the equations of motion. However, upon coupling to the scalar field, he is no longer a full derivative, and as such should add new terms to the field equations.

Appendix A Notations

Flat space-time

$$\eta = diag(-1,1,1,1) \ . \tag{A.1}$$

Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{A.2}$$

. Ricci tensor

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \ . \tag{A.3}$$

Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} \ . \tag{A.4}$$

Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ . \tag{A.5}$$

Kretschmann scalar

$$\mathcal{K} = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \ . \tag{A.6}$$

The electromagnetic tensor

$$F = \partial_\mu A_\nu - \partial_\mu A_\nu \ . \tag{A.7}$$

D'alembertian

$$\Box\phi = g^{\mu\nu}\phi_{;\mu\nu} \ . \tag{A.8}$$

Appendix B Variation of the Action

This section contains an explicit variation of the action (2.1) with respect to the metric. In the variation of the first term I will mainly follow the derivation from [4] (our notations are different, so the result will slightly differ). Starting with the first term:

$$\begin{aligned}\delta S_I &= \delta\left(-\frac{1}{16\pi}\int d^4x\sqrt{-g}\phi R\right) = -\frac{1}{16\pi}\delta\left(\int d^4x\sqrt{-g}\phi g^{\mu\nu}R_{\mu\nu}\right)\\ &= \delta S_{\sqrt{-g}} + \delta S_\phi + \delta S_{g^{\mu\nu}} + \delta S_{R_{\mu\nu}}\ .\end{aligned}$$

Variation of the measure will give us

$$\delta S_{\sqrt{-g}} = -\frac{1}{16\pi}\int d^4x\delta(\sqrt{-g})\phi g^{\mu\nu}R_{\mu\nu} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(-\frac{1}{2}\right)\phi g_{\mu\nu}R\delta g^{\mu\nu}\ .$$

The second part contribution is zero by definition. The third is given by

$$\delta S_{g^{\mu\nu}} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\phi R_{\mu\nu}\delta g^{\mu\nu}\ .$$

Now comes the tricky part. the variation of Ricci tensor is

$$\delta S_{R_{\mu\nu}} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\phi\left[\nabla_\mu\nabla_\nu(\delta g^{\mu\nu}) - \nabla_\sigma\nabla^\sigma(g_{\mu\nu}\delta g^{\mu\nu})\right]\ .$$

In case of Einstein GR, ϕ is constant, so this term is a full derivative. Thus we can use Stoke's theorem to turn it to a vanishing integral at infinity. In case ϕ is not a constant we can integrate by parts twice to get

$$\delta S_{R_{\mu\nu}} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi_{;\mu\nu} - g_{\mu\nu}g^{\alpha\beta}\phi_{;\alpha\beta}\right)\delta g^{\mu\nu}\ .$$

The second term in action (2.1) will simply give the contribution from $\sqrt{-g}$, which is just the potential multiplied by $-(1/2)g_{\mu\nu}$.

We can now turn to the third (EM) term in 2.1.

$$\delta S_{III} = \delta\left(-\frac{1}{4}\int d^4x\sqrt{-g}F^2\right) = \delta S_{\sqrt{-g}} + \delta S_{F^2}\ . \tag{B.1}$$

The first part is the same as before

$$\delta S_{\sqrt{-g}} = -\frac{1}{4}\int d^4x\sqrt{-g}\left(-\frac{1}{2}\right)g_{\mu\nu}F^2\delta g^{\mu\nu}\ .$$

recalling that $F^2 = g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}$ the second term (after relabeling the indices and contracting the remaining $g^{\mu\nu}$) is

$$\delta S_{F^2} = -\frac{1}{2}\int d^4x\sqrt{-g}F_{\mu}{}^{\alpha}F_{\nu\alpha}\delta g^{\mu\nu} \ .$$

Putting it all together, we can finally write down the last field equation:

$$\phi G_{\mu\nu} = -\phi_{;\mu\nu} + g_{\mu\nu}\Box\phi + \frac{3}{4a}\left(\phi - \frac{1}{G}\right)^2 g_{\mu\nu} - 8\pi F_{\mu}{}^{\alpha}F_{\nu\alpha} + 2\pi F^2 g_{\mu\nu}. \tag{B.2}$$

Same as (2.4).

Appendix C Manipulation of the Field Equation

In this appendix we will see how to get Eqs. (2.10)-(2.12) from eq. (2.4). Let $B_{\mu\nu} = 0$ be the general form of the field equations, i.e. equation (2.4) with all the terms are moved to the LHS. The first equation will be given by $e^{\lambda(r)-\nu(r)}B_{tt} + B_{rr} = 0$ and could be rearranged to

$$\nu'(r) = \frac{2\phi''(r)}{\left(\frac{2}{r}\phi(r) + \phi'(r)\right)} - \lambda'(r) \ . \tag{C.1}$$

The second equation will be given by $e^{\lambda(r)-\nu(r)}B_{tt} - B_{rr} = 0$

$$\phi''(r) + \left(\frac{2}{r} + \frac{\nu'(r) - \lambda'(r)}{2}\right)\phi'(r) - \frac{e^{\lambda(r)}}{aG}\left(\phi(r) - \frac{1}{G}\right) = 0 \ . \tag{C.2}$$

The last field equation is given by

$$e^{\lambda(r)-\nu(r)}B_{tt} - B_{rr} - \frac{2}{r^2}e^{\lambda(r)}\left[B_{\theta\theta} - \frac{1}{2}\left(R + \frac{3}{a}\left(\phi - G^{-1}\right)\right)r^2\phi(r)\right] \ .$$

notice that the extra term is actually zero (equation 2.2). The last field equation is therefore

$$\begin{aligned}\phi''(r) \quad &+\frac{1}{2}\left(\lambda'(r) - \nu'[r]\right)\left(\frac{2}{r}\phi(r) - \phi'(r)\right) - \frac{2}{r^2}\left(1 - e^{\lambda(r)}\right)\phi(r) \\ &-\frac{3e^{\lambda(r)}}{2a}\left(\phi(r) - \frac{1}{G}\right)\left(\phi(r) + \frac{1}{3G}\right) - 8\pi\frac{e^{\lambda(r)}\tilde{Q}^2}{r^4} = 0 \ .\end{aligned} \tag{C.3}$$

Appendix D Matching Boundary Conditions

We show here the approximation done in section 3.3 in order to obtain Eqs. (3.30)-(3.31).

A careful analysis of the numerical solution (both the asymptotic and the interior) lead us to the following conclusions (see figure 14):

1. $e^{\lambda}\left(\phi - \frac{1}{G}\right) \to 0$,
2. $\phi \to \frac{1}{G}$,
3. $r \to h$,
4. $\frac{\phi'}{\phi} \to 0$.

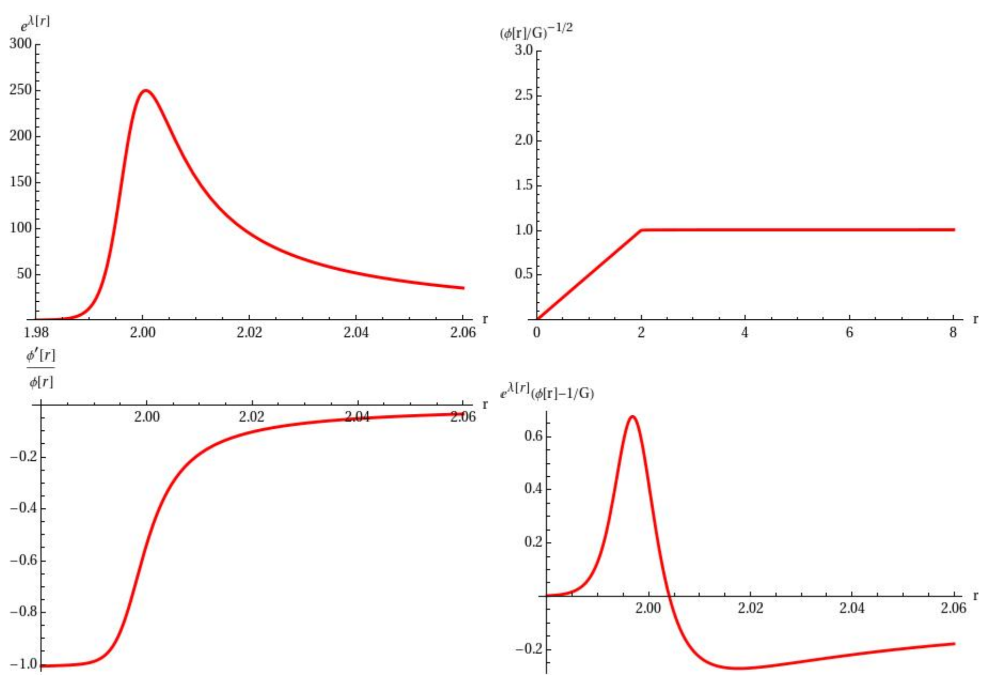

Figure 14: Numerical evaluation for some of the terms in the modified field equations

Using the first assumption, equation (3.3) turns to be

$$\phi''(r) + \left(\frac{2}{r} + \frac{\nu'(r) - \lambda'(r)}{2}\right)\phi'(r) = 0 \ , \tag{D.1}$$

which we can integrate once to obtain our first "horizon" equation,

$$\phi'(r) r^2 e^{\frac{\lambda(r)-\nu(r)}{2}} = A \ . \tag{D.2}$$

Our next equation will be a combination of two equations:

$$\frac{r}{2\phi}\left((3.3)-(3.4)\right) = \frac{\phi'(r)}{\phi(r)} + \frac{\nu'(r)-\lambda'(r)}{2} + \frac{1}{r} = \frac{e^{\lambda(r)}}{r}\left(1-\frac{Q^2}{r^2\phi(r)}\right) \ , \tag{D.3}$$

under our assumptions this equation turns to be

$$\nu'(r) - \lambda'(r) = \frac{2}{h} e^{\lambda(r)}\left(1-\frac{GQ^2}{h^2}\right) \ . \tag{D.4}$$

For the last equation, we will take eq. (3.2) and rewrite it as

$$\nu'(r) + \lambda'(r) = \frac{2\phi''(r)}{\frac{2}{r}\phi(r) + \phi'(r)} \ . \tag{D.5}$$

Using our last two derived equations (D.2) and (D.4) for the R.H.S of the equation to get

$$\lambda'(r) + \nu'(r) = -\frac{\frac{2}{h} A e^{\lambda(r)}\left(1-\frac{GQ^2}{h^2}\right) e^{\frac{\nu(r)-\lambda(r)}{2}}}{2\frac{h}{G} + A e^{\frac{\nu(r)-\lambda(r)}{2}}} \ . \tag{D.6}$$

We now introduce a new parameter σ:

$$\sigma(r) = \frac{\lambda(r)-\nu(r)}{2} \ , \tag{D.7}$$

so we can rewrite equations (D.4) and (D.6) as

$$\sigma'(r) = -\frac{1}{h} e^{\lambda(r)}\left(1-\frac{GQ^2}{h^2}\right) \ , \tag{D.8}$$

$$\frac{\sigma''(r)}{\sigma'(r)} - \sigma'(r) = \frac{\sigma'(r) e^{\sigma(r)}}{\frac{2h}{AG} + e^{\sigma(r)}} \ . \tag{D.9}$$

Integrating (D.9) once will give us

$$\sigma'(r) = \frac{p}{h}\left(\frac{2h}{AG} + e^{\sigma(r)}\right) e^{\sigma(r)} \ , \tag{D.10}$$

and again

$$r - \bar{r} = -\frac{h}{p}\left(\frac{AG}{2h}\right)\left(e^{-\sigma(r)} + \frac{AG}{2h}\mathrm{Log}\left(\frac{1}{1+\frac{2h}{AG}e^{-\sigma(r)}}\right)\right) \ , \tag{D.11}$$

where p and $\bar{r}$ are constants of integration. From comparing (D.8) with (D.10) we can get

$$e^{\lambda(r)} = -\frac{pe^{\sigma(r)}}{1-\frac{GQ^2}{h^2}}\left(\frac{2h}{AG}+e^{\sigma(r)}\right) \ , \tag{D.12}$$

$$e^{\nu(r)} = e^{\lambda(r)}e^{-2\sigma(r)} = -\frac{pe^{-\sigma(r)}}{1-\frac{GQ^2}{h^2}}\left(\frac{2h}{AG}+e^{\sigma(r)}\right) \ . \tag{D.13}$$

Eqs. (D.11), (D.12) and (D.13) form our solution for the *would-have-been horizon.* It can be simplified further more by introducing a new parameter x:

$$x = \frac{2h}{AG}e^{-\sigma(r)} \ . \tag{D.14}$$

The asymptotic solution will respond to $x \to \infty$ and the core solution to $x \to -1^+$. The above equations can be rewritten as

$$r - \bar{r} = -\frac{h}{p}\left(\frac{AG}{2h}\right)^2 (x - \mathrm{Log}[1+x]) \ , \tag{D.15}$$

$$e^{\lambda(r)} = -\frac{p}{1-\frac{GQ^2}{h^2}}\left(\frac{2h}{AG}\right)^2\left(\frac{1+x}{x^2}\right) \ , \tag{D.16}$$

$$e^{\nu(r)} = -\frac{p}{1-\frac{GQ^2}{h^2}}(1+x) \ . \tag{D.17}$$

List of Figures

Bibliography

[1] A. Davidson and I. Gurwich, "Hollowgraphy Driven Holography: Black Hole with Vanishing Volume Interior," *Int. J. Mod. Phys.*, vol. D19, pp. 2345–2351, 2010.

[2] A. Davidson and I. Gurwich, "Holographic entropy packing inside a black hole," *Phys. Rev. Lett.*, vol. 106, p. 151301, Apr 2011.

[3] A. Davidson and B. Yellin, "Spontaneously Induced General Relativity: Holographic Interior for Reissner-Nordstrom Exterior," *Submitted to Phys. Rev. D*, 2011.

[4] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Benjamin Cummings, 2003.

[5] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* Wiley, first edition ed., July 1972.

[6] R. M. Wald, *General Relativity.* University Of Chicago Press, first edition ed., June 1984.

[7] P. K. Townsend, "Black holes," 1997.

[8] S. F. Ross, "Black hole thermodynamics," 2005.

[9] R. M. Wald, "The thermodynamics of black holes," *Living Rev.Rel.*, vol. 4, p. 6, 2001.

[10] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Communications in Mathematical Physics*, vol. 31, pp. 161–170, June 1973.

[11] C. Brans and R. H. Dicke, "Mach's principle and a relativistic theory of gravitation," *Phys. Rev.*, vol. 124, pp. 925–935, Nov 1961.

[12] P. J. Steinhardt and F. S. Accetta, "HYPEREXTENDED INFLATION," *Phys. Rev. Lett.*, vol. 64, p. 2740, 1990.

[13] A. A. Sen, S. Sen, and S. Sethi, "Dissipative fluid in brans-dicke theory and late time acceleration," *Phys. Rev. D*, vol. 63, p. 107501, Apr 2001.

[14] T. P. Sotiriou and V. Faraoni, "f(R) Theories Of Gravity," *Rev. Mod. Phys.*, vol. 82, pp. 451–497, 2010.

[15] A. Bhadra, K. Sarkar, D. P. Datta, and K. K. Nandi, "Brans-Dicke theory: Jordan vs Einstein frame," *Mod. Phys. Lett.*, vol. A22, pp. 367–376, 2007.

[16] V. Faraoni and E. Gunzig, "Einstein frame or Jordan frame?," *Int. J. Theor. Phys.*, vol. 38, pp. 217–225, 1999.

[17] E. E. Flanagan, "The conformal frame freedom in theories of gravitation," *Class. Quant. Grav.*, vol. 21, p. 3817, 2004.

[18] R. Arnowitt, S. Deser, and C. W. Misner, "Coordinate invariance and energy expressions in general relativity," *Phys. Rev.*, vol. 122, p. 997, May 1961.

[19] A. Komar, "Positive-definite energy density and global consequences for general relativity," *Phys. Rev.*, vol. 129, pp. 1873–1876, Feb 1963.

[20] S. W. Hawking, "Gravitational radiation from colliding black holes," *Physical Review Letters*, vol. 26, pp. 1344–1346, May 1971.

[21] J. D. Bekenstein, "Black holes and entropy," *Physical Review D*, vol. 7, pp. 2333–2346, Apr. 1973.

[22] S. Hawking, "Particle creation by black holes," *Communications in Mathematical Physics*, vol. 43, pp. 199–220, Aug. 1975.

[23] L. Smarr, "Mass formula for Kerr black holes," *Phys.Rev.Lett.*, vol. 30, pp. 71–73, 1973.

[24] G. 't Hooft, "Dimensional reduction in quantum gravity," 1993. Essay dedicated to Abdus Salam.

[25] R. Bousso, "The holographic principle," *Rev. Mod. Phys.*, vol. 74, pp. 825–874, Aug 2002.

[26] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.*, vol. 2, pp. 231–252, 1998.