Three-dimensional recurring patterns in excitable media

Y. Biton, A. Rabinovitch, D. Braunstein, M. Friedman, I. Aviram

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
Physics Department, Sami Shamoon College of Engineering, Beer-Sheva, Israel
Department of Information Systems Engineering, Ben-Gurion University, Beer-Sheva 84105, Israel

Abstract

A new method to create three-dimensional periodic patterns in excitable media is presented. The method is demonstrated and the patterns are obtained with the help of two types of 3D "spiral pairs" generators, which are respectively based on a "corner effect" and a "unidirectional propagation" processes. The results portray time-repeating patterns resembling fruits or potteries. The method is easy to implement and can be used to form other types of 3D patterns in excitable media. The question of periodicity of the patterns thus obtained is resolved by calculating the singular lines (filaments) around which they evolve. Both types of spiral pairs are easily adaptable to different excitable media. These methods of generating scroll waves (or, in 2D, spiral pairs) are quite different from the known Winfree's techniques. Our method which is based on the vulnerable window (VW) approach, is demonstrated and the patterns are obtained with the help of two types of 3D "spiral pairs" generators, (a) a mechanism based on the "corner effect" and (b) a procedure based on unidirectional propagation. These methods of generating scroll waves (or, in 2D, spiral pairs) are quite different from the known Winfree's technique which is based on the vulnerable window (VW) approach. A typical study of the VW mechanism is given in [25]; See also [26] in which the method is more rigorously discussed. Our methods are basically spatial while the VW one is temporal. Thus, in the unidirectional method the dependence on the timing of the

1. Introduction

Excitable media are ubiquitous in many areas. They appear in the heart [1] and in the Cajal [2] and nervous [3] systems of the body; in Belousov–Zhabotinsky and related chemical reactions [4]; in morphological biology [5]. Three-dimensional (3D) patterns (TDP) appearing in excitable media have been studied both experimentally and theoretically [6,7]. Following in the footsteps of Winfree [8,9], these patterns include either scroll waves, which are 3D extensions of 2D spirals, or regular scroll rings (see e.g. [10–14] for descriptions and mathematical analyses) which are closed scroll waves, and can even be interlaced. Such patterns have recently been found experimentally in the Belousov–Zhabotinsky reaction [15]. 3D steady state (Turing) patterns were obtained numerically in the FitzHugh–Nagumo and Brusselator systems [16]. Genuine time-dependent 3D patterns appear in ventricular fibrillation (VF) of the heart [17] but their actual production is not completely understood. Since numerical calculation of genuine 3D patterns is still very demanding and time consuming, the full expance of their potential has only barely been touched upon. In this Letter we numerically describe the production and development of novel 3D oscillating patterns, i.e. patterns developing around a central core and by periodically returning to the original site of incipience are being recreated time and again. Note that the difference between our patterns and regular scroll rings is that the latter, being attached to an expanding or a shrinking filament, are unstable and eventually disappear, unless generated with a very specific set of parameters allowing shape retention [18]. Our patterns, on the other hand, are periodic which makes them quite different, namely they are related to a pulsating, ring shaped filament. Additionally, as expected, these TDP's simultaneously serve as sources of outgoing quasi-spherical waves. Typically, experimentally observed regular scroll rings tend to shrink and disappear. This case is termed [19] 'positive filament tension'. Expanding regular scroll rings, or filaments having negative tension, have also been observed in the Belousov–Zhabotinsky reaction [20]. These were proposed as the main cause of turbulence in the heart, leading to ventricular fibrillation and sudden death [19,21], although they were found only in low-excitability media different from cardiac tissue. In a recent publication [22], such negative tension scroll rings were numerically shown to exist in high-excitability media–media actually resembling the heart tissue. There are several ways to generate these patterns. Here we describe two three-dimensional methods which are adaptations of our previous 2D techniques: (a) a mechanism based on the "corner effect" [23], and (b) a procedure based on unidirectional propagation [24]. These methods of generating scroll waves (or, in 2D, spiral pairs) are quite different from the known Winfree's technique which is based on the vulnerable window (VW) approach. A typical study of the VW mechanism is given in [25]; See also [26] in which the method is more rigorously discussed. Our methods are basically spatial while the VW one is temporal. Thus, in the unidirectional method the dependence on the timing of the
two stimulations is of secondary importance. In fact, in most of the works (the present one included) which use this method, the two stimulations are applied simultaneously. Furthermore, in the corners method, a single stimulation is sufficient to induce spiral-pairs (scroll-waves). It is demonstrated that the extension of these methods to 3D is feasible, although, of course, with different parameter values. Moreover, it is demonstrated that periodic patterns can be realized in 3D as well, and this periodicity is connected to a novel behavior of the patterns filaments.

Note that it is not the method of generation of scroll rings in 3D or spiral-pairs in 2D that we want to emphasize here, but the generation of their sources, namely the creation of entities which repeatedly generate such patterns. One can probably obtain such sources also by the Winfree methods of spiral pair (or scroll ring) generation (e.g. the VW procedure) by choosing the parameter values so that when the arms coalesce, they would leave behind an un-annihilated part that can regrow into a new pair (ring). We adopted the methods presented here since source generation by them is easy and straightforward.

2. Generation and methods

We use the FitzHugh–Nagumo (FHN) system of differential equations to characterize the medium and the dynamics:

\[
\frac{\partial v}{\partial t} = D \nabla^2 v + v(v-a)(1-v) - w + \delta(t) I(x,y,z)
\]

\[
\frac{\partial w}{\partial t} = \varepsilon(v-d \cdot w)
\]

(1)

All variables in Eq. (1) are dimensionless. Here \(v(x,y,z,t)\) is an activator variable (action potential in the case of the neural or cardiac electricity), while \(w(x,y,z,t)\) is an inhibitor, \(D\) is the diffusion coefficient, \(a, d\) and \(\varepsilon\) are constants controlling the shape of the propagating waves of \((v, w)\); \(\nabla^2\) is the three-dimensional Laplacian. \(\delta(t) I(x,y,z)\) is a short input current (stimulation). The system is maintained in its excitable mode: There is a value of the current amplitude \(I(x,y,z,t)\) (delivered within a specified area) that acts as a threshold. Only a stimulation amplitude above this value leads to a propagating wave.

We shall now briefly describe the above mentioned two ways of generating time periodic spiral pairs, and the method employed to obtain the singular lines.

The corner effect

Consider an initial \((t = 0)\) cube in which a stimulation is applied to a 3D excitable medium (Fig. 1(a)). The stimulation is applied in the form of a short current surge just above threshold, and only inside the cube. Throughout the entire medium the excitability remains the same at all times. Due to the high curvature at the corners of the cube, the outgoing wave generated by the stimulation cannot emerge there, giving rise to six finite waves, i.e. having free edges. These start to bend inwards forming cap-like features and, if conditions are right, when annihilating during mutual encounter, leave behind an undamaged part, a part which becomes the source of the next generation of waves (for more details of the generation process, albeit for a 2D case, see [23]).

The unidirectional propagation method

Consider a 3D stimulation arrangement (Fig. 2(a)). It consists of a small cube adjacent to a larger one (the smaller cube appears just below the larger one in Fig. 2(a)). At a certain time the small and the large cubes are stimulated by the above- and the below-threshold currents, respectively. Under appropriate conditions (similar to but different than those of [24]) a spiral pair source is created by the inwards rotation of the generated unidirectional wave (Fig. 2) in a manner somewhat similar to that of the corner effect.

Singular lines

In a 3D physical space, singular lines (usually called filaments) [8,10] appear. In a sense they constitute the generators around which the patterns evolve. Finding the singular lines is relatively simple when the controlling equation is known (as is presently the case), since only a single singular point has to be obtained in an elaborate manner: A 3D grid of points is built in the spatial region in which the three-dimensional patterns are to be created. The equations of motion (1) are solved under the initial conditions described above (corners or unidirectional). The solution of the system is composed of two functions \(v(t)\) and \(w(t)\) which the phase plane \((v, w)\) passing through the origin \(v = w = 0\). A singular point is a position \(T\) for which the above mentioned closed loop \(w(T)\) shrinks to a point \((v_0, w_0)\) and the phase with respect to this point, defined by \(\psi = \arctan \left(\frac{w(T)}{v(T)}\right)\) is undetermined. At a certain time say \(t'\), there are points \(F\) around which the phase \(\psi\) (defined with respect to any point \(v_0, w_0\) inside the \(w(T)\) curve there) takes all values in the range \((0, 2\pi)\) [27,28]. The values \(v_1, w_1\) at such a point are those of singular points throughout. The most frequent pair is taken as the best approximation of...
and 0.1572. Figs. 1(b)–(f) display the temporal evolution of the range for this cube is found to be between 0.1567 length 40 centered about the origin and a stimulating amplitude equipotential surfaces serve as sources for the next generation of waves. Video films exhibit the full dynamics are available [29].

3. Results and discussion

We solved the three-dimensional FHN equation within a 3D grid of 120 × 120 × 120 points using explicit central finite-difference numerical method. In the following numerical calculations we used the values $D = 0.19$, $a = 0.12$, $d = 3$, $\varepsilon = 0.005$. Simulations were stopped following 60,000 time units. The space and time steps were $\Delta x = \Delta y = \Delta z = 2$, and $\Delta t = 0.2$. A finer grid, $\Delta x = \Delta y = \Delta z = 0.5$ and $\Delta t = 0.125$ was also employed and it was found to lead to practically the same results. In particular, no meandering and only a very slight change in the period of the system were found under the finer grid. The 3D corner effect was achieved by just-above-threshold stimulation of a cube within an excitable medium (Fig. 1(a), $t = 0$). Each pair of orthogonal sides of the cube forms a 90° corner. We used a stimulated cube of side length 40 centered about the origin and a stimulating amplitude of 0.1569 (the threshold amplitude being 0.1555). The stimulating amplitude range for this cube is found to be between 0.1567 and 0.1572. Figs. 1(b)–(f) display the temporal evolution of the equipotential surfaces $v(x, y, z) = 0.5$. Note that in all figures, spatial coordinates are given in length units. As can be seen from Fig. 1(b), the cube edges disappear leaving six “caps” and a hollow region. As time increases, the caps grow in size, rotate inwards and coalesce, each leaving a viable perpetual source of waves. (Not seen!) See Figs. 1(e) and (f). The structures inside the envelope serve as sources for the next generation of waves. Video films exhibiting the full dynamics are available [29].

$v_1$, $w_1$. For the parameters given below these were $v_1 = -0.054$, $w_1 = 0.0002$. Other singular points at any time $t$ can be obtained by searching for all points $\mathbf{r}$ for which $v$ and $w$ are equal to these values. They comprise the filaments.

For the unidirectional propagation, two adjacent cubes are stimulated simultaneously (Fig. 2(a), $t = 0$); the small cube, of side length 10, by a stimulating amplitude of 0.2 (above-threshold) and the large cube, of side 40 by a stimulating amplitude of 0.1550 (below-threshold). For these side length and stimulating amplitude of the large cube, the minimum side length and amplitude required for the small one are: 8 and 1 = 0.2, respectively. For the same side length of the small cube, the stimulating amplitude for the large cube is between 0.150 and 0.1555. The range of parameters of the FHN system for which the TDP can be created is given in Table 1.

Table 1: The range of FHN parameters for creating the TDP.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Range value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>0.13 &lt; $D$ &lt; 0.23</td>
</tr>
<tr>
<td>$a$</td>
<td>0.1180 &lt; $a$ &lt; 0.1216</td>
</tr>
<tr>
<td>$d$</td>
<td>2.3 &lt; $d$ &lt; 3.7</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>0.0046 &lt; $\varepsilon$ &lt; 0.0052</td>
</tr>
</tbody>
</table>

Fig. 2 presents, for the unidirectional propagation, a sequence of snapshots taken at successive times. (Video films are available in [30].) They show the complete evolution of the 3D patterns. The amplitude of $v$ in the small cube expands, while diminishing in the large cube. Consequently, $v$ of the small cube envelopes that of the larger one. Fig. 2(b) shows the equipotential ($v = 0.5$) surface after 100 time units. The initial large cube has vanished and the small cube’s $v$ has developed into a bowl-shaped structure. As time increases, Figs. 2(c) and (d), the upper rim rotates inwards, closes on itself and forms an “apple” shaped feature (Fig. 2(d)). Similar to the corner-effect case, a source of perpetual waves is left inside the “apple” and serves as an active generator for the ensuing waves (Figs. 2(e) and (f)).

Figs. 3 and 4 display the equipotential surfaces $v(x, y, z) = 0.5$ (Figs. 3(a) and 4(a)), and the corresponding filaments (Figs. 3(b) and 4(b)) at a specific time. Fig. 3 relates to the corner mechanism
while Fig. 4 corresponds to the unidirectional mechanism. Fig. 3 is drawn at the time $t = 282$ and Fig. 4 at $t = 520$. The filaments were calculated in the manner described in Section 2. As can be seen from Fig. 3(b), the filaments consist of six ring-like lines originating near the edges of the original stimulated cube.

The filament displayed in Fig. 4(b) exhibits a similar behavior to that of Fig. 3(b). However, it is situated inside the active source seen in Fig. 4(a). Note that the filaments are not exact circles. Each of them does not even stay within one plane.

We now follow the filaments in time. Fig. 5 displays a cross section of the TDP for the unidirectional mechanism in the plane $x = 0$ at four different times. The projection of the filament, onto this plane is marked by a black broken line whose ends are the filament interceptions with the plane. As seen, the patterns show a unique property. They are built around a closed singular line (filament), similar to the case of a scroll ring. However, the uniqueness of our structures, which, in contrast with the ephemeral regular scroll rings, consists of two features, both related to the possibility of the recurrent appearance of a source of the new configuration. The first feature is that in our method the temporal development proceeds from the outer parts of the spirals towards their cores. Generally, for regular scroll rings, movement proceeds from the cores outwards. The second feature is that the patterns are “not complete” scroll rings. A complete scroll ring is one in which the arms of spirals of opposite sides of the ring are not touching at or near the core. They do touch when the ring evolves, but at their outer parts. The spiral arms in our patterns on the other hand meet (near the cores) before evolving into complete rings. This encounter is the ingredient that provides the possibility of the source creation and thereby the possibility of producing the repeating (periodic) patterns.

A different type of “oscillations” which appear in dual diffusion scroll rings (Ref. [10], specifically Fig. 10 and related discussion) are transient phenomena and can last for some time before ring disappearance. They are different from the present stable periodic patterns.

Under certain conditions regular scroll rings could lead to the generation of repeating patterns. This situation would occur in the situation that, as the arms of the ring from all sides meet, they encounter in such a way as to leave behind a viable 3D source. However, in this case the new repeating pattern (similar to the ones generated by our method, including the progress towards the core) will dwell at a position different from that of the initial ring. To our knowledge, no such scenarios have been reported in the literature.

How is this routine reflected in the filament behavior? As can be seen from Fig. 5, the filament only contracts somewhat during the time interval when the ingoing surfaces of the structure get closer, but does not disappear completely; and when the source is created, the filament gets attached to it (in a continuous way) and re-stretches as the source turns into the new structure. This seemingly recurring behavior continues during a very long time. The periodic behavior of the filament is depicted in Fig. 6. An approximate radius of the (almost circular) filament is calculated and shown to increase by almost twice from its smallest value, just prior to the source creation, to its largest value, at the most open structure. For the corner effect case, each of the six filaments behaves in a similar way (not shown).

4. Summary

A new tool to construct 3D patterns in excitable media has been presented. Its performance was demonstrated by two different types of periodic textures. This new approach offers the possibility to create rich diversity of forms. The examples shown here have either cubic or $C_4$ symmetries and additional shapes with different symmetries and beautiful structures can easily be gener-
The structures are shown to behave differently than regular scroll rings. While it is well known [31] that a regular scroll ring either collapses or extends, in line with its filament tension, $\alpha$. Its radius $R$ changes [32] according to: $\frac{dR}{dt} = -\alpha/R$. Our patterns on the other hand show “scroll rings” of a periodic nature, where no real $\alpha$ applies and the time dependence of $R$ is approximately related to its second derivative.

The new TDP’s can possibly be observed experimentally in chemical reactions such as the Belousov–Zhabotinsky.

It was assumed [19,21] that regular scroll-rings, even though being of limited duration, were one of the main sources of fibrillation in the heart. They were assumed to be easily broken by interacting with the boundaries of or obstacles in the tissue, thus leading to the detrimental turbulence. Several works [33,34] suggested external means (such as fluctuations or fields) to prolong their duration. Being unconditionally persistent, the present TDP’s, if they should appear in the heart, could lead to even more complicated arrhythmia than the ephemeral regular scroll rings.

References

[30] http://physics.bgu.ac.il/~yaacov/ALP2box.wmv. Note that the video is composed of several time intervals. The initial part spans the interval between 0 and 720 time units. The next parts span the time interval approximately between 4000 and 5000 time units.